NEWCOMB'S MATHEMATICAL COURSE

KEY

TO

NEWCOMB'S

COLLEGE ALGEBRA

NEW YORK
HENRY HOLT AND COMPANY
1882

ELECTROTYPED AND PRINTED BY
S. W. GREEN'S SON,
74 and 76 Beekman Street,
NEW YORK.

PREFACE.

THE greater part of this Key has been prepared, and the proofs read, by Professors J. Howard Gore, of Columbian University, Washington, and J. W. Gore, of the University of Virginia. It is modelled so closely after the familiar form as scarcely to call for remark. A possible exception is offered in the case of Book X., where the forms of reasoning applicable to problems in the Combinatory Analysis are given in full.

A few oral exercises on the principles taught in the opening book have been added for the practice of beginners in the subject.

KEY TO NEWCOMB'S COLLEGE ALGEBRA.

§ 26.

1. $4 + 12 = 16$. $\quad -6 - 1 - 18 = -25$.

$\therefore 16 - 25 = -9$. Ans.

2. $-6 - 3 - 8 = -17$. Ans.

3. 34. $\quad -6 - 10 - 9 = -25$.

$\therefore 34 - 25 = 9$. Ans.

4. -17, sum in Ex. 2.
9, sum in Ex. 3.
-26. Ans.

5. $5 + 3 = 8$. $\quad -6 - 1 - 16 = -23$.

$\therefore 8 - 23 = -15$.

$+8$. $\quad -2 - 7 - 4 = -13$.

$8 - 13 = -5$.

Then $-5 - (-15) = 10$. Ans.

6. $5 + 3 = 8$. $\quad -6 - 1 - 16 = -23$.

$8 - 23 = -15$.

$7 + 4 = 11$. $\quad -3 - 8 = -11$.

$11 - 11 = 0$.

Then $-15 - 0 = -15$. Ans.

7. $-7 \times 8 = -56$. Ans.

8. $-8 \times 7 = -56$. Ans.

9. $6 \times 7 = 42$. $\quad -5 \times -4 = 20$.

$\therefore 42 \times 20 = 840$. Ans.

10. $-6 \times -11 = 66) \times 8 = 528) \times -2 = -1056$. Ans.

11. $-1 \times -1 = 1$. $\quad -1 \times -1 = 1$.

$1 \times 1 = 1$. Ans.

12. 9, sum in Ex. 3.
-9, sum in Ex. 1.
18, diff. $\quad \therefore 18 \times -17 = -306$. Ans.

17. $= 4^2 . 9 + (3^2 . -5) =$
$16 . 9 + 9 . -5.$
$16 \times 9 = 144.$
$9 \times -5 = -45. \qquad \therefore 144 - 45 = 99.$ Ans.

18. $= 3^3 . 4 - (-7 . 9^2) =$
$27 . 4 - (-7 . 81).$
$27 \times 4 = 108.$
$-7 \times 81 = -567. \qquad \therefore 108 - (-567) = 675.$ Ans.

19. $= -7^2 + (-5^2) =$
$49 + 25 = 74.$ Ans.

20. $= -7^3 + (-5^3) =$
$-343 - 125 = -468.$ Ans.

21. $= -7^3 - (-5^3) =$
$-343 - (-125) = -218.$ Ans.

22. $-7^3 . 3 . -(-5^3 . 4) =$
$-343 . 3 - (-125 . 4).$
$-343 \times 3 = -1029.$
$-125 \times 4 = -500.$
$\therefore (-1029) - (-500) = -529.$ Ans.

23. $= -7^3 . -5^2 - (3^3 . 4^2) =$
$-343 . 25 - (27 . 16).$
$-343 \times 25 = -8575.$
$27 \times 16 = 432.$
$\therefore -8575 - 432 = -9007.$ Ans.

24. $= (-7^2 . -5^3) - (-5^2 . 3^3) =$
$(49 . -125) - 25 . 27.$
$49 \times -125 = -6125.$
$25 \times 27 = 675.$
$\therefore -6125 + (-675) = -6800.$ Ans.

25. $= (-7 . -5^2) + (-7^2 . -5) =$
$(-7 . 25) + (49 . -5)$
$-7 \times 25 = -175.$
$49 \times -5 = -245.$
$\therefore -175 + (-245) = -420.$ Ans.

26. $(-7 . -5^3) - (-7^3 . -5) =$
$(-7 . -125) - (-343 . -5).$
$-7 \times -125 = +875.$
$-343 \times -5 = +1715.$
$\therefore 875 - 1715 = -840.$ Ans.

27. $= \dfrac{(-7.-5)+3.4}{(-7.-5)-3.4} = \dfrac{-7\times -5+3\times 4}{-7\times -5-3\times 4} =$

$\dfrac{35+12}{35-12} = \dfrac{47}{23} = 2\tfrac{1}{23}$. **Ans.**

28. $= \dfrac{(-7.0)-(-5.9)}{(-5.4)-(3.9)} = \dfrac{-7\times 0-(-5\times 9)}{-5\times 4-3\times 9} =$

$\dfrac{0-(-45)}{-20-27} = -\dfrac{45}{47}$. **Ans.**

29. $= \dfrac{2.3^2.4^2-10.3^3}{9-(-5.0.3)} = \dfrac{2\times 9\times 16-10\times 27}{9-(-5\times 0\times 3)} =$

$\dfrac{288-270}{9-0} = \dfrac{18}{9} = 2$. **Ans.**

30. $= \dfrac{-7.-5-(3.9)}{3-4} = \dfrac{-7\times -5-(3\times 9)}{3-4} =$

$\dfrac{35-27}{3-4} = \dfrac{8}{-1} = -8$. **Ans.**

31. When $x = -7$,
$= (-7)^2 + (-3.-7) + 8 =$
$49 + (-3\times -7) + 8$.
$49 + 21 + 8 = 78$. **Ans.**
When $x = -6$,
$= (-6)^2 + (-3.-6) + 8 =$
$36 + (-3\times -6) + 8$.
$36 + 18 + 8 = 62$. **Ans.**
When $x = -5$,
$= (-5)^2 + (-3.-5) + 8 =$
$25 + (-3\times -5) + 8$.
$25 + 15 + 8 = 48$. **Ans.**
When $x = -4$,
$= (-4)^2 + (-3.-4) + 8 =$
$16 + (-3\times -4) + 8$.
$16 + 12 + 8 = 36$. **Ans.**
When $x = -3$,
$= (-3)^2 + (-3.-3) + 8 =$
$9 + (-3\times -3) + 8$.
$9 + 9 + 8 = 26$. **Ans.**
When $x = -2$,
$= (-2)^2 + (-3.-2) + 8 =$
$4 + (-3\times -2) + 8$.
$4 + 6 + 8 = 18$. **Ans.**

When $x = -1$,
$= (-1)^2 + (-3 \,.\, -1) + 8.$
$1 + (-3 \times -1) + 8.$
$1 + 3 + 8 = 12.$ Ans.

When $x = 0$,
$= 0^2 + (-3 \,.\, 0) + 8.$
$0 + (-3 \times 0) + 8.$
$0 + 0 + 8 = 8.$ Ans.

When $x = 1$,
$= 1^2 + (-3 \,.\, 1) + 8 =$
$1 + (-3 \times 1) + 8.$
$1 - 3 + 8 = 6.$ Ans.

When $x = 2$,
$= 2^2 + (-3 \,.\, 2) + 8 =$
$4 + (-3 \times 2) + 8.$
$4 - 6 + 8 = 6.$ Ans.

When $x = 3$,
$= 3^2 + (-3 \,.\, 3) + 8 =$
$9 + (-3 \times 3) + 8.$
$9 - 9 + 8 = 8.$ Ans.

When $x = 4$,
$= 4^2 + (-3 \,.\, 4) + 8 =$
$16 + (-3 \times 4) + 8.$
$16 - 12 + 8 = 12.$ Ans.

When $x = 5$,
$= 5^2 + (-3 \,.\, 5) + 8 =$
$25 + (-3 \times 5) + 8.$
$25 - 15 + 8 = 18.$ Ans.

When $x = 6$,
$= 6^2 + (-3 \,.\, 6) + 8 =$
$36 + (-3 \times 6) + 8.$
$36 - 18 + 8 = 26.$ Ans.

When $x = 7$,
$= 7^2 + (-3 \,.\, 7) + 8 =$
$49 + (-3 \times 7) + 8.$
$49 - 21 + 8 = 36.$ Ans.

32. When $x = -7$,

$$= \frac{8 + (-3 \,.\, -7)}{8 - (-3 \,.\, -7)} = \frac{8 + (-3 \times -7)}{8 - (-3 \times -7)} = \frac{8 + 21}{8 - 21} =$$

$$-\frac{29}{13}. \quad \text{Ans.}$$

When $x = -6$,

$$= \frac{8 + (-3 \,.\, -6)}{8 - (-3 \,.\, -6)} = \frac{8 + (-3 \times -6)}{8 - (-3 \times -6)} = \frac{8 + 18}{8 - 18} = -\frac{26}{10}. \quad \text{Ans.}$$

When $x = -5$,

$$= \frac{8 + (-3 \,.\, -5)}{8 - (-3 \,.\, -5)} = \frac{8 + (-3 \times -5)}{8 - (-3 \times -5)} = \frac{8 + 15}{8 - 15} = -\frac{23}{7}. \quad \text{Ans.}$$

When $x = -4$,

$$= \frac{8 + (-3 \,.\, -4)}{8 - (-3 \,.\, -4)} = \frac{8 + (-3 \times -4)}{8 - (-3 \times -4)} = \frac{8 + 12}{8 - 12} = \frac{20}{-4} = -5. \quad \text{Ans.}$$

When $x = -3$,

$$= \frac{8 + (-3 \,.\, -3)}{8 - (-3 \,.\, -3)} = \frac{8 + (-3 \times -3)}{8 - (-3 \times -3)} = \frac{8 + 9}{8 - 9} = \frac{17}{-1} = -17. \quad \text{Ans.}$$

When $x = -2$,

$$= \frac{8 + (-3 \,.\, -2)}{8 - (-3 \,.\, -2)} = \frac{8 + (-3 \times -2)}{8 - (-3 \times -2)} = \frac{8 + 6}{8 - 6} = \frac{14}{2} = 7. \quad \text{Ans.}$$

When $x = -1$,

$$= \frac{8 + (-3 \,.\, -1)}{8 - (-3 \,.\, -1)} = \frac{8 + (-3 \times -1)}{8 - (-3 \times -1)} = \frac{8 + 3}{8 - 3} = \frac{11}{5}. \quad \text{Ans.}$$

When $x = 0$,

$$= \frac{8 + (-3 : 0)}{8 - (-3 \,.\, 0)} = \frac{8 + (-3 \times 0)}{8 - (-3 \times 0)} = \frac{8 + 0}{8 - 0} = \frac{8}{8} = 1. \quad \text{Ans.}$$

When $x = 1$,

$$= \frac{8 + (-3 \,.\, 1)}{8 - (-3 \,.\, 1)} = \frac{8 + (-3 \times 1)}{8 - (-3 \times 1)} = \frac{8 - 3}{8 - (-3)} = \frac{5}{11}. \quad \text{Ans.}$$

When $x = 2$,

$$= \frac{8 + (-3\,.\,2)}{8 - (-3\,.\,2)} = \frac{8 + (-3 \times 2)}{8 - (-3 \times 2)} = \frac{8 + (-6)}{8 - (-6)} =$$

$$\frac{8 - 6}{8 + 6} = \frac{2}{14}. \quad \text{Ans.}$$

When $x = 3$,

$$= \frac{8 + (-3\,.\,3)}{8 - (-3\,.\,3)} = \frac{8 + (-3 \times 3)}{8 - (-3 \times 3)} = \frac{8 + (-9)}{8 - (-9)} =$$

$$\frac{8 - 9}{8 + 9} = -\frac{1}{17}. \quad \text{Ans.}$$

When $x = 4$,

$$= \frac{8 + (-3\,.\,4)}{8 - (-3\,.\,4)} = \frac{8 + (-3 \times 4)}{8 - (-3 \times 4)} = \frac{8 + (-12)}{8 - (-12)} =$$

$$\frac{8 - 12}{8 + 12} = -\frac{4}{20}. \quad \text{Ans.}$$

When $x = 5$,

$$= \frac{8 + (-3\,.\,5)}{8 - (-3\,.\,5)} = \frac{8 + (-3 \times 5)}{8 - (-3 \times 5)} = \frac{8 + (-15)}{8 - (-15)} =$$

$$\frac{8 - 15}{8 + 15} = -\frac{7}{23}. \quad \text{Ans.}$$

When $x = 6$,

$$= \frac{8 + (-3\,.\,6)}{8 - (-3\,.\,6)} = \frac{8 + (-3 \times 6)}{8 - (-3 \times 6)} = \frac{8 + (-18)}{8 - (-18)} =$$

$$\frac{8 - 18}{8 + 18} = -\frac{10}{26}. \quad \text{Ans.}$$

When $x = 7$,

$$= \frac{8 + (-3\,.\,7)}{8 - (-3\,.\,7)} = \frac{8 + (-3 \times 7)}{8 - (-3 \times 7)} = \frac{8 + (-21)}{8 - (-21)} =$$

$$\frac{8 - 21}{8 + 21} = -\frac{13}{29}. \quad \text{Ans.}$$

§ 43.

When $x = -3$,

1. $$= \frac{-3(-3+1)(-3+2)(-3+3)}{1 \times 2 \times 3 \times 4} =$$

$$\frac{(+9-3)(-3+2)(-3+3)}{1 \times 2 \times 3 \times 4} = \frac{6 \times -1 \times 0}{1 \times 2 \times 3 \times 4} =$$

$$\frac{0}{24}. \quad \text{Ans.}$$

When $x = -1$,

$$= \frac{-1(-1+1)(-1+2)(-1+3)}{1 \times 2 \times 3 \times 4} =$$
$$\frac{(1-1)(-1+2)(-1+3)}{1 \times 2 \times 3 \times 4} = \frac{0 \times 1 \times 2}{1 \times 2 \times 3 \times 4} =$$
$$\frac{0}{24}. \quad \text{Ans.}$$

When $x = 1$,

$$= \frac{1(1+1)(1+2)(1+3)}{1 \times 2 \times 3 \times 4} = \frac{(1+1)(1+2)(1+3)}{1 \times 2 \times 3 \times 4} =$$
$$\frac{2 \times 3 \times 4}{1 \times 2 \times 3 \times 4} = \frac{24}{24} = 1. \quad \text{Ans.}$$

When $x = 3$,

$$= \frac{3(3+1)(3+2)(3+3)}{1 \times 2 \times 3 \times 4} = \frac{(9+3)(3+2)(3+3)}{1 \times 2 \times 3 \times 4} =$$
$$\frac{12 \times 5 \times 6}{1 \times 2 \times 3 \times 4} = \frac{360}{24} = 15. \quad \text{Ans.}$$

2. When $x = -3$,

$$= \frac{[-1(3+3) - 3(-1+3)]^2}{5(3+3) + 3(5+3)} =$$
$$\frac{[(-3-3) - (-3+9)]^2}{15+15+15+9} = \frac{(-6-6)^2}{30+24} = \frac{-12^2}{54} =$$
$$\frac{144}{54}. \quad \text{Ans.}$$

When $x = -1$,

$$= \frac{[-1(3+1) - 3(-1+1)]^2}{5(3+1) + 3(5+1)} =$$
$$\frac{[-3-1-(-3+3)]^2}{15+5+15+3} = \frac{[-4-0]^2}{20+18} = \frac{-4^2}{38} =$$
$$\frac{16}{38}. \quad \text{Ans.}$$

When $x = 1$,

$$= \frac{[-1(3-1) - 3(-1-1)]^2}{5(3-1) + 3(5-1)} =$$
$$\frac{[-3+1-(-3-3)]^2}{15-5+15-3} = \frac{[-2-(-6)]}{30-8} =$$
$$\frac{[-2+6]^2}{22} = \frac{4^2}{22} = \frac{16}{22}. \quad \text{Ans.}$$

When $x = 3$.

$$= \frac{[-1(3-3) - 3(-1-3)]^2}{5(3-3) + 3(5-3)} =$$

$$\frac{[(-3+3) - (-3-9)]^2}{15 - 15 + 15 - 9} = \frac{[0 - (-12)]^2}{30 - 24} =$$

$$\frac{12^2}{6} = \frac{144}{6} = 24. \quad \text{Ans.}$$

3. When $x = -3$,

$$= [(-1.-3) + 3(-3+1)^2 + 5(-3+1)^3]^3 \frac{-3-5}{-3+5} =$$

$$[3 + 3(-2)^2 + 5(-2)^3]^3 \frac{-8}{2} =$$

$$[3 + 3(4) + 5(-8)]^3 (-4) =$$

$$[3 + 12 - 40]^3 (-4) = -25^3 \times -4 =$$

$$-15625 \times -4 = 62500. \quad \text{Ans.}$$

When $x = -1$,

$$= [(-1.-1) + 3(-1+1)^2 + 5(-1+1)^3]^3 \frac{-1-5}{-1+5} =$$

$$[1 + 3(0)^2 + 5(0)^3]^3 \frac{-6}{4} = (1 + 0 + 0)^3 \frac{-6}{4} =$$

$$1 \times \frac{-6}{4} = -\frac{6}{4}. \quad \text{Ans.}$$

When $x = 1$,

$$= [(-1.1) + 3(1+1)^2 + 5(1+1)^3]^3 \frac{1-5}{1+5} =$$

$$[-1 + 3(2)^2 + 5(2)^3]^3 \frac{-4}{6} =$$

$$[-1 + 3(4) + 5(8)]^3 \frac{-4}{6} =$$

$$(-1 + 12 + 40)^3 \frac{-4}{6} = 51^3 \times \frac{-4}{6} = -\frac{530604}{6}. \quad \text{Ans.}$$

When $x = 3$,

$$= [(-1.3) + 3(3+1)^2 + 5(3+1)^3]^3 \frac{3-5}{3+5} =$$

$$[-3 + 3(4)^2 + 5(4)^3]^3 \frac{-2}{8} =$$

$$[-3 + 3(16) + 5(64)]^3 \frac{-2}{8} =$$

$$(-3 + 48 + 320)^3 \frac{-2}{8} = (365)^3 \frac{-2}{8} =$$

$$-\frac{97254250}{8}. \quad \text{Ans.}$$

4. When $x = -3$,

$$= [\sqrt{(5.-3^2+3)} - \sqrt{(5-3^2-3)}]\sqrt{(5.3+1)} =$$
$$[\sqrt{(5.9+3)} - \sqrt{(5.9-3)}]\sqrt{(5.3+1)} =$$
$$[\sqrt{(45+3)} - \sqrt{(45.-3}]\sqrt{(15+1)} =$$
$$(\sqrt{48} - \sqrt{42})\sqrt{16} = (\sqrt{48} - \sqrt{42})\,4. \quad \text{Ans.}$$

When $x = -1$,

$$= [\sqrt{(5.-1^2+3)} - \sqrt{(5-1^2-3)}]\sqrt{(5.3+1)} =$$
$$[\sqrt{(5.1+3)} - \sqrt{(5.1-3)}]\sqrt{(5.3+1)} =$$
$$[\sqrt{(5+3)} - \sqrt{(5-3)}]\sqrt{(15+1)} =$$
$$(\sqrt{8} - \sqrt{2})\sqrt{16} = (\sqrt{8} - \sqrt{2})\,4. \quad \text{Ans.}$$

When $x = 1$,

$$= [\sqrt{(5.1^2+3)} - \sqrt{(5.1^2-3)}]\sqrt{(5.3+1)} =$$
$$[\sqrt{(5.1+3)} - \sqrt{(5.1-3)}]\sqrt{(5.3+1)} =$$
$$[\sqrt{(5+3)} - \sqrt{(5-3)}]\sqrt{(15+1)} =$$
$$(\sqrt{8} - \sqrt{2})\sqrt{16} = (\sqrt{8} - \sqrt{2})\,4. \quad \text{Ans.}$$

When $x = 3$,

$$= [\sqrt{(5.3^2+3)} - \sqrt{(5.3^2-3)}]\sqrt{(5.3+1)} =$$
$$[\sqrt{(5.9+3)} - \sqrt{(5.9-3)}]\sqrt{(5.3+1)} =$$
$$[\sqrt{(45+3)} - \sqrt{(45-3)}]\sqrt{(15+1)} =$$
$$(\sqrt{48} - \sqrt{42})\sqrt{16} = (\sqrt{48} - \sqrt{42})\,4. \quad \text{Ans.}$$

§ 48.

1. $a + bx - (x - y)$.

2. $x - y - (a + bx)$.

3. $a + bx - \left(\dfrac{a - bx}{m}\right)$.

4. $\dfrac{a - bx}{m} - mpq$.

5. $\sqrt{(a + bx)}$.

6. $\sqrt{(a + bx + x - y)}$.

7. $\sqrt{[a + bx - (x - y)]}$.

8. $(a + bx)^2 (x - y)^2$.

9. $(mpq)^2$.

10. $(x-y)^2 (mpq)^2$.

11. $$\frac{mpq\,(a+bx) - \dfrac{a-bx}{m}(x-y)}{\left(\dfrac{a-bx}{m}\right)^2 - (x-y)^2}.$$

12. $$\frac{(a+bx)(x-y)}{\left(\dfrac{a-bx}{m}\right) mpq}.$$

13. $$\frac{[a+bx+x-y]\,[a+bx-(x-y)]}{\left(\dfrac{a-bx}{m}+mpq\right)\left(\dfrac{a-bx}{m}-mpq\right)}.$$

14. $$\frac{[3\,(a+bx)-2\,(x-y)]^2}{\left[4\left(\dfrac{a-bx}{m}\right)\right]^2}.$$

15. $$\frac{(a+bx)^2-(x-y)}{\sqrt{[a+bx-(x-y)]^2}}.$$

16. $$\frac{2\,(a+bx+x-y)^2}{[2\,(x-y)-mpq]^2}.$$

17. $$\frac{(a+bx)\,[(x-y)^2-mpq]}{(a+bx)^2\,(x-y)^2}.$$

18. $$\frac{\left(\dfrac{a-bx}{m}\right)^2-(x-y)^2}{(a+bx)(x-y)+(x-y)\dfrac{a-bx}{m}}.$$

19. $$\frac{\sqrt{2\,(a+bx)}+2\sqrt{(x-y)}}{(a+bx+x-y)^2}.$$

20. $$\frac{(a+bx)^2+\left(\dfrac{a-bx}{m}\right)^2}{[mpq-(x-y)]\,[mpq+x-y]}.$$

21. $$\left[\left(mpq+\frac{a-bx}{m}\right)^2+a+bx\right](x-y).$$

22. $$\frac{(a+bx)\sqrt{\left[\left(\frac{a-bx}{m}\right)^2-(x-y)\right]}}{\left[\sqrt{(a+bx)}-\frac{a-bx}{m}\right]^2}.$$

23. $$\frac{mpq-\left[\left(\frac{a-bx}{m}\right)^2-(x-y)\right]^2}{a+bx-\left[\frac{a-bx}{m}+x-y\right]\left[\frac{a-bx}{m}-(x-y)\right]}.$$

24. $$\frac{mpq-\left(\frac{a-bx}{m}\right)^2+x-y}{a+bx-\left[\left(\frac{a-bx}{m}+mpq\right)(x-y)\right]}.$$

1. As there are 100 cents in one dollar, there are 100 times m cents in m dollars. Hence, $100\,m$. Ans.

2. Since there are 100 cents in a dollar, in m cents there will be as many dollars as 100 is contained in m.

$$\frac{m}{100}.\quad \text{Ans.}$$

3. His a dollars are $100\,a$ cents. Hence total number of cents, $100\,a+b$.

His b cents are $\frac{b}{100}$ dollars. Hence he has $a+\frac{b}{100}$ dollars.

4. $(a+b)\,m$,
$[(a+b)\,m]^2$.

5. $b-(m+n)$ or $b-m-n$.

6. In one dollar there are 100 cents.
In m dollars " " $100\,m$ cents.
$\therefore\ \frac{100\,m}{k}=$ number of chickens which can be bought.

7. $\frac{m}{4}=$ time it takes him to go, and

$\frac{m}{3}=$ " " " " " come;

then $\frac{m}{4}+\frac{m}{3}=$ time to go and return.

8. $h + k =$ cost of one peck of each;
since there is the same number of pecks, of each $(h + k)$ $\times$ number of pecks $= m$;

then $\dfrac{m}{h+k} =$ number of pecks.

9. b miles an hour is equal to $\dfrac{b}{60}$ miles per minute. To go a miles will require $\dfrac{a}{\frac{b}{60}}$ minutes, or $\dfrac{60\,a}{b}$.

10. (1) $ax = $ cost of tea,
$by =$ " " sugar,
$cz =$ " " coffee,

$ax + by + cz =$ entire cost in cents.

(2) 100 cents $=$ number of cents in a dollar;
then $\dfrac{ax + by + cz}{100} =$ entire cost in dollars.

(3) $10 =$ number of mills in 1 cent;
then $(ax + by + cz)\,10 =$ entire cost in mills.

11. There are 100 cents in 1 dollar;
then $100\,x =$ number of cents in x dollars.
$fm =$ cost of flour,
$\therefore 100\,x - fm =$ change he is to receive.

12. $\dfrac{a}{m} =$ time it takes first to walk the distance, and

$\dfrac{a}{n} =$ " " " second " " " "

$\therefore$ since they walk towards each other it will take $\dfrac{a}{m+n}$ hours for them to meet.

(2) First travels at m miles an hour.

$\therefore m\left(\dfrac{a}{m+n}\right) =$ distance he will have gone.

Second travels at n miles an hour.

$\therefore n\left(\dfrac{a}{m+n}\right) =$ distance he will have gone.

13. ax = value of bonds in dollars,
by = " " land " "
$ax + by$ = " " estate " "
mq = amount he owed.
Then $ax + by - mq$ = remainder after payment.
$\therefore \frac{ax + by - mq}{n}$ = share of each child.

14. $\frac{(x+y)s}{a+b}$, $k - \frac{(x+y)s}{a+b}$.

15. $\frac{p+q}{a+b}$ $\frac{p-q}{a-b}$,
$\left[\frac{p+q}{a+b} - \frac{p-q}{a-b}\right](r+s)$.

16. $\left(\frac{y}{b} - \frac{x}{a}\right)\frac{x+y}{x-y}$.

17. $\frac{(x+6)(a+b)+q}{r-s}$.

18. $a \times a \times 1000$ = value of houses in dollars;
100 cents in 1 dollar;
then $(a \times a) \times 1000 \times 100$ = value in cents.

19. ax = value of a lbs. at x cents a lb.,
by = " " b " " y " " "
then $ax + by$ = ent. value, and $a + b$ = the no. of lbs.
$\therefore \frac{ax + by}{a+b}$ = value of mixture.

20. $(x+y)(a+b)$ = number of rooms.
$(x+y)(a+b)(m+n)$ = no. of pieces of furniture.

21. $(p+q)(p+q)$ = number of pages, and
$(p+q)(p+q)(p+q)$ = number of words.
$\therefore (p+q)(p+q)(p+q)\,8$ = number of letters.

22. $3k$ = distance traversed by first before second started,
and $3k + kx$ = whole distance traversed by first.
mx = " " " " second.
$\therefore 3k + kx - mx$ = distance first is ahead of second.

23. $\frac{m}{r}$ = number of hours it takes first man;

$\frac{m}{s}$ = " " " " " second man;

then $\frac{m}{r} - \frac{m}{s}$ = number of hours first will arrive ahead.

(2) $\frac{m}{s} - \frac{m}{r}$ = " " " second " " "

24. Since first runs n miles an hour for n hours,
$n \times h$ = distance between the cities.
The second will run $n + 5$ miles an hour,
$\therefore \frac{nh}{n+5}$ = time of second train.

25. $\frac{t}{h}$ = cost of one horse;

$\frac{m}{n}$ = " " " yoke of oxen.

$\therefore \frac{t}{h} - \frac{m}{n}$ = cost of horse exceeds that of yoke of oxen,

and $\frac{m}{n} - \frac{t}{h}$ = cost of yoke of oxen exceeds that of horse.

26. $\frac{1}{2}$ of $2m$ miles = m miles;

$\frac{m}{r}$ = time to make first half;

$\frac{m}{s}$ = " " " second half;

then $\frac{m}{r} + \frac{m}{s}$ = time to make the distance,

and $\therefore \dfrac{2m}{\frac{m}{r} + \frac{m}{s}}$ = average speed.

27. Distance being a miles, B, going q miles an hour, will reach New Haven in $\frac{a}{q}$ hours

In this time A, going p miles an hour, will make $p\frac{a}{q}$ miles.

Subtracting the a miles to Hartford we have left $p\frac{a}{q} - a$ as his distance on return journey.

28. $6b =$ cost of books;
then $k - 6b =$ amount left.
$\therefore \dfrac{k - 6b}{4} =$ number of books which can be bought.

29.

$$\begin{aligned} as &= \text{cost of sugar;} \\ br &= \text{`` `` coffee;} \\ \hline as + br &= \text{entire cost in cents.} \end{aligned}$$

100 cents in 1 dollar;

then $\dfrac{as + br}{100} =$ entire cost in dollars.

$m - \dfrac{as + br}{100} =$ amount left.

$\therefore \dfrac{m - \dfrac{as + br}{100}}{q} =$ no. bbls. flour which can be bought.

30. $\dfrac{m}{a} =$ what each Chinee received;

$\dfrac{n}{h} =$ " " orphan "

then $\dfrac{2m}{a} + \dfrac{3n}{h} =$ amount expended in bibles;

and $\therefore \dfrac{\dfrac{2m}{a} + \dfrac{3n}{h}}{x} =$ cost of each Bible.

31. $mk =$ number of miles he travels in k hours;
then $a - mk =$ number of miles to be travelled;
and $h - k =$ number of hours remaining.
$\therefore \dfrac{a - mk}{h - k} =$ rate to travel remaining distance.

32. $rk =$ number of miles it ran in k hours;
then $x - rk =$ number of miles to be travelled;

and $\left(k + \dfrac{m}{60}\right)$ hours $=$ time since it has started;

hence $h - \left(k + \dfrac{m}{60}\right) =$ time to travel remainder,

and $\dfrac{x - rk}{h - \left(k + \dfrac{m}{60}\right)} =$ rate for remainder of journey.

§ 54.

1.
$$\begin{array}{rrrrr} & 3a + 7b & - 8c & + d & \\ & 3a - 2b & + c & & - e \\ - & a - b & - c & - d & \\ \hline & 5a + 4b & - 8c & & - e \end{array}$$ Ans.

2.
$$\begin{array}{rl} 7a - & (x+y) \\ 8a - & (x+y) \\ -16a + 3 & (x+y) \\ \hline - a + & (x+y) \end{array}$$ Ans.

3.
$$\begin{array}{r} 7x^2 - 2x - 5 \\ 2x^2 - 3x + 8 \\ -9x^2 + 5x + 3 \\ \hline 6 \end{array}$$ Ans.

4.
$$\begin{array}{r} x^2 + 2x - y \\ 4x^2 + 7x - 2y \\ -2x^2 + x - 9y \\ -3x^2 - x - y \\ \hline 9x - 13y \end{array}$$ Ans.

5.
$$\begin{array}{l} 9(a+b)^2 \\ 10 \\ 1 \\ 2 \\ \qquad\qquad - x - y - z \\ \hline 22(a+b)^2 - x - y - z \end{array}$$ Ans.

6.
$$\begin{array}{rr} 2(m+n) & + 3(a+b) \\ -1 & +1 \\ -1 & +1 \\ \hline & 5(a+b) \end{array}$$ Ans.

7.
$$\begin{array}{r} 7a^3 - 2a^2 + 3ax \\ - a^3 - a^2 - ax \\ -6a^3 + 3a^2 - 2ax \\ \hline 0 \end{array}$$ Ans.

8.
$$\begin{array}{lrr} (m+n)^2 & + x & \\ 2 & & - y \\ 3 & - 2x & \\ 1 & & - y \\ \hline 7(m+n)^2 & - x & - 2y \end{array}$$ Ans.

9.
$$\begin{array}{llllll}
 & (p+q)^2 - 6 & & & \\
1 & & +a & & \\
1 & & & +b & \\
1 & & & & +c \\
\hline
4 & (p+q)^2 - 6 & +a & +b & +c
\end{array}$$
Ans.

10.
$$\begin{array}{l}
6a(x-y) \\
5a \\
2a \\
a \\
\hline
14a(x-y)
\end{array}$$
Ans.

11.
$$\begin{array}{llr}
2(m-n)x & & +\ 2 \\
 & +\ 3(m+n)x & -\ 5 \\
 & +\ 5 & -\ 6 \\
 & +\ 7 & -\ 8 \\
\hline
2(m-n)x & +\ 15(m+n)x & -\ 17
\end{array}$$
Ans.

12.
$$\begin{array}{lll}
3\frac{x}{a} & & \\
2 & +3\frac{y}{b} & \\
1 & -1 & \\
 & +1 & -\frac{6}{7} \\
1 & & -\frac{1}{7} \\
\hline
7\frac{x}{a} & +3\frac{y}{b} & -1
\end{array}$$
Ans.

13.
$$\begin{array}{lll}
\frac{x}{y} & - & \frac{m}{n} \\
2 & - & 2 \\
3 & - & 3 \\
4 & - & 4 \\
\hline
10\frac{x}{y} & - & 10\frac{m}{n}
\end{array}$$
Ans.

14.
$$\begin{array}{l}
\frac{x+y}{m+n} \\
3 \\
5 \\
7 \\
\hline
16\frac{x+y}{m+n}
\end{array}$$
Ans.

15. $2x - 3y$ what first had; $\quad\therefore\quad 2x - 3y$
$\underline{x - y}$ " second " more; $\quad 3x - 4y$
$3x - 4y$ " " " $\quad \overline{5x - 7y}$ both. **Ans.**

16.
$$\begin{aligned} A &= 2x \\ B &= 2x - y \\ C &= 4x + y \\ \hline \text{All} &= 8x \end{aligned}$$
Ans.

17.
$$\begin{aligned} \text{Eldest} &= x \\ \text{Third} &= x - 5 \\ \text{Second} &= x - 10 \\ \text{Young} &= x - 15 \\ \hline \text{All} &= 4x - 30 \end{aligned}$$
Ans.

§ 55.

1. $(a + m)x + (b + n)y.$

2. $= (mn + pq)x + (2b - 4b)y = (mn + pq)x - 2by.$

3. $= (3 + 6b + 7a)x + (-2 - 4)y + m + n =$
$(3 + 6b + 7a)x - 6y + m + n.$

4. $= (8a + 8b + 7 + 1)x + (b - 5 - 5)y =$
$(8a + 8b + 8)x + (b - 10)y =$
$8(a + b + 1)x + (b - 10)y.$

5. $(a - m)x + (b - n)y + (c - p)z.$

6. $(2d - 2f)x + (3e - 3d)y + (4f + 4e)z.$

7. $(\frac{2}{3}a + \frac{3}{4}b)y + (6a - 2)x.$

8. $(2a - 3b)x + (-4a - b)y.$

9. $(\frac{1}{2}a - \frac{1}{6}m)x + (\frac{2}{3}b + \frac{3}{4}n)y.$

10. $= (4m - 3a - 6c - \frac{2}{3}m + \frac{1}{2}d)x + (2 + a)y.$
But $4m - \frac{2}{3}m = \frac{12}{3}m - \frac{2}{3}m = \frac{10}{3}m.$
$\therefore (\frac{10}{3}m - 3a - 6c + \frac{1}{2}d)x + (2 + a)y.$

11. $= (5ab - ab - d)x + (4cd - 3mn)y =$
$(4ab - d)x + (4cd - 3mn)y.$

12. $= (2b - \frac{1}{4}d - 3b)x + (3a + 2a)y =$
$(-b - \frac{1}{4}d)x + 5ay.$

13. $= (-3-5)x + (\frac{1}{2}a + 2 - \frac{3}{4}a + 1)y.$
But $\frac{1}{2}a - \frac{3}{4}a = \frac{2}{4}a - \frac{3}{4}a = -\frac{1}{4}a.$
$\therefore\ -8x + (-\frac{1}{4}a + 3)y.$

14. $(3m - a + 1 + d)x + (-\frac{1}{2}a - 1)y.$

15. $3abx + (-m - d)y + (2c + 1)\sqrt{x}.$

16. $= -6x - y - 3\sqrt{x} + (5m + 4 + 1)\sqrt{y} =$
$-6x - y - 3\sqrt{x} + (5m + 5)\sqrt{y}.$

17. $= cx - 6y + (4 + 1)\sqrt{x} + (a - 1 - 4a)\sqrt{y} =$
$cx - 6y + 5\sqrt{x} + (-3a - 1)\sqrt{y}.$

§ 56.

5.
$$\begin{array}{l} 8a + 14b \\ 6a + 20b \\ \hline 2a - 6b \quad \text{Ans.} \end{array}$$

6.
$$\begin{array}{l} a - b + c - d \\ -a + b - c + d \\ \hline 2a - 2b + 2c - 2d \quad \text{Ans.} \end{array}$$

7.
$$\begin{array}{l} 8a - 2b + 3c \\ 4a - 6b - c - 2d \\ \hline 4a + 4b + 4c + 2d \quad \text{Ans.} \end{array}$$

8.
$$\begin{array}{l} 2x^2 - 8x - 1 \\ 5x^2 - 6x + 3 \\ \hline -3x^2 - 2x - 4 \quad \text{Ans.} \end{array}$$

9.
$$\begin{array}{l} 4x^4 - 3x^3 - 2x^2 - 7x + 9 \\ x^4 - 2x^3 - 2x^2 + 7x - 9 \\ \hline 3x^4 - x^3 \qquad - 14x + 18 \quad \text{Ans.} \end{array}$$

10.
$$\begin{array}{l} 2x^2 - 2ax + 3a^2 \\ x^2 - ax + a^2 \\ \hline x^2 - ax + 2a^2 \quad \text{Ans.} \end{array}$$

11.
$$\begin{array}{l} a^3 - 3a^2b + 3ab^2 - b^3 \\ -a^3 + 3a^2b \\ \hline 2a^3 - 6a^2b + 3ab^2 - b^3 \quad \text{Ans.} \end{array}$$

12.
$$\begin{array}{l} 7x^3 - 2x^2 + 2x + 2 \\ 4x^3 - 2x^2 - 2x - 14 \\ \hline 3x^3 \qquad + 4x + 16 \quad \text{Ans.} \end{array}$$

13.
$$\begin{array}{l} \quad 5(x-y)+7(x-z)+9(z-x) \\ \quad 9 \qquad\quad\; +7 \qquad\quad\; +5 \\ \hline -4(x-y) \qquad\qquad\quad +4(z-x) \end{array} \quad \text{Ans.}$$

14.
$$\begin{array}{l} 12(a-b)-3(a+b)+7a-2b \\ \;\;7 \qquad\quad -5 \\ \hline 5(a-b)+2(a+b)+7a-2b \end{array} \quad \text{Ans.}$$

15.
$$\begin{array}{l} \quad 7\frac{x}{y}-11\frac{y}{z}-15\frac{z}{x} \\ -5 \quad +6 \quad -7 \quad +8\frac{a}{b} \\ \hline 12\frac{x}{y}-17\frac{y}{z}-8\frac{z}{x}-8\frac{a}{b} \end{array} \quad \text{Ans.}$$

§ 58.

1. $=x-y+x+y=2x.$

2. $=x+y+y-x=2y.$

3. $=3ab-2mp+ab-3x-2mp=$
$4ab-4mp-3x.$

4. $=2ax-3by+mx-2ax-pz+3by=$
$mx-pz.$

5. $3\frac{a}{b}+\frac{a}{b}-2\frac{m}{n}+\frac{a}{b}+2\frac{m}{n}=5\frac{a}{b}.$

§ 59.

1. $=ab-m+3ab-2ax-7ab=$
$-3ab-m-2ax.$

2. $=x-a+x+x-a=3x-2a.$

3. $=2b+b-2c-b-2c=2b-4c.$

4. $=4x-3y+2z+7x-5y+3z-x+y=$
$10x-7y+5z.$

5. $=7ax-2by-8ax-3by-8ax+3by$
$-9ax-2by.$

6. $=a-x-a-x+2x=0.$

7. $= -a + b - b + c - c + a = 0.$

8. $= -3m - 2n - 3m + 2n + 9m = 3m.$

§ 60.

2. $x - (m + n).$

4. $x - (3b - 2c - 5d).$

6. $x - (-x - a + b).$

7. $x - (-2x + 2m - 2n).$

8. $x - (-2x - ab + m + 3ab - 2m) =$
$x - (-2x + 2ab - m).$

10. $x - (-3 + a + b).$

11. $x - (-a + b - c - m + n).$

12. $x - (am + b + p - q + am - n) =$
$x - (2am + b + p - q - n).$

13. $x - (a + b + p - q + m - n).$

§ 61.

1. $= m + [-p + q + a - b - c + d] =$
$m - p + q + a - b - c + d.$

2. $= m - [-a + b - p - q + n - k] =$
$m + a - b + p + q - n + k.$

3. $= 7ax - [2ax + by - 3ax + by - 7ax + 2by] =$
$7ax - 2ax - by + 3ax - by + 7ax - 2by =$
$15ax - 4by.$

4. $= a - [a - \{a - [a - a + a]\}] =$
$a - [a - \{a - a + a - a\}] =$
$a - [a - a + a - a + a] =$
$a - a + a - a + a - a = 0.$

5. $= p - [a - b - s - t - a - m - n] =$
$p - a + b + s + t + a + m + n =$
$p + b + s + t + m + n.$

6. $= 2ax - [3ax - by - 7ax - 2by - 5ax + 3by] =$
$2ax - 3ax + by + 7ax + 2by + 5ax - 3by = 11ax.$

7. $= ax + by + cz + [2ax - 3cz - 2cz - 5ax - 7by + 3cz] =$
$ax + by + cz + 2ax - 3cz - 2cz - 5ax - 7by + 3cz =$
$-2ax - 6by - cz.$

8. $= x - \{2x - y - [3x - 2y - 4x + 3y]\} =$
$x - \{2x - y - 3x + 2y + 4x - 3y\} =$
$x - 2x + y + 3x - 2y - 4x + 3y = -2x + 2y.$

9. $= ax - bz - \{ax + bz - [ax - bz - ax - bz]\} =$
$ax - bz - \{ax + bz - ax + bz + ax + bz\} =$
$ax - bz - ax - bz + ax - bz - ax - bz = -4bz.$

10. $= my - \{x + 3y + [2my - 3x + 3y - 4ab] + 5\} =$
$my - \{x + 3y + 2my - 3x + 3y - 4ab + 5\} =$
$my - x - 3y - 2my + 3x - 3y + 4ab - 5 =$
$-(m + 6)y + 2x + 4ab - 5.$

11. $= ax + 4cx - mx - cx + y + [mx - cx - y] =$
$ax + 4cx - mx - cx + y + mx - cx - y = ax + 2cx.$

12. $= 3ax - 3bx + 3ay + 3az - 3by - 3bz =$
$3(a - b)x + 3(a - b)y + 3(a - b)z.$

13. $= 13ax + 2xy - d - [7ad + xy + d] - 4xy =$
$13ax + 2xy - d - 7ad - xy - d - 4xy =$
$13ax - 3xy + (-2 - 7a)d.$

14. $= m + 4x - [-4y + 2x + ay - x + p] =$
$m + 4x + 4y - 2x - ay + x - p =$
$m + 3x + (4 - a)y - p.$

15. $= 2a\sqrt{y} - 3m - [b\sqrt{x} - 6n + \sqrt{y} - 2\sqrt{y}] =$
$2a\sqrt{y} - 3m - b\sqrt{x} + 6n - \sqrt{y} + 2\sqrt{y} =$
$(2a + 1)\sqrt{y} - 3m - b\sqrt{x} + 6n.$

§ 69.

2. $6a^2bx^3$. Ans.

3. $15m^4xy$. Ans.

4. $42a^2m^2y$. Ans.

5. $4a^2m^2$. Ans.

6. $5x^3y^6z^2$. Ans.

7. $9x^2y^2z^2$. Ans.

8. $4a^2b^2m^2$. Ans.

9. $9a^4b^4x^4$. Ans.

10. $2.6\,mpqr = 12\,mpqr.$
$2.6\,pqrs = 12\,pqrs.$ $\therefore 12\,mpqr \times 12\,pqrs = 144\,mp^2q^2r^2s.$

11. $144ax^2y^2z$. Ans.

12. $m^{11}x^6y^7$. Ans.

13. $3n^2k^2m$. Ans.

14. $14abcd^2efg$. Ans.

§ 70.

15. m^3xyz. Ans.
16. $abcdx^4$. Ans.
17. $12\,a^2b^2m^2n^2$. Ans.
18. $14\,a^2b^2c^3$. Ans.
19. $135\,m^3n^3p^3$. Ans.
20. $6\,a^4bcdmxy^3z^2$. Ans.
21. $a^6m^3n^2x^6y^2z$. Ans.
22. $a^{10}x^7y^7$. Ans.
23. $48\,a^4m^4n^2x^3$. Ans.

§ 72.

1. a^4bcdm. Ans.
2. $-abcdx^4$. Ans.
3. $-a^3b^2cx^4$. Ans.
4. $30\,a^6b^3mx^2$. Ans.
5. $105\,a^3m^2xy^3$. Ans.
6. $10\,n^7x^{m+n}yz^2$. Ans.
7. $4\,abmn$. Ans.
8. $168\,abkm^2x^3$. Ans.
9. $6\,bgmny^3$. Ans.
10. $4\,ax^4y^6$. Ans.
11. $-30\,agx^2y^3z^3$. Ans.
12. $15\,a^2b^3nx^3yz$. Ans.
13. $-4\,abgxyz^4$. Ans.
14. $4\,bc^2gnx^2z^5$. Ans.
15. $-3\,ab^3e^3x^2y$. Ans.
16. $4\,abexy$. Ans.
17. $-24\,a^4x^3y^3$. Ans.
18. $a^4x^4y^3$. Ans.
19. $-3\,a^4x^3y^3$. Ans.
20. $-m^7n^4x^3$. Ans.
21. $a^5bx^4y^3$. Ans.
22. $-apqx^4y^3$. Ans.
23. $3\,a^3bcd^2x^3$. Ans.
24. $9\,acm^2x^3y^2$. Ans.
25. $-\frac{2}{3}\,acm^3n^2x^3$. Ans.
26. $3\,a^3bcxy^3$. Ans.
27. $-a^8bdx^5$. Ans.
28. $-30\,a^2m^4n^4y$. Ans.
29. $m^2n^2x^3y$. Ans.
30. $-\frac{1}{3}\,m^2pqx^3y^3$. Ans.

§ 73.

1.
$$\begin{array}{r} 3\,x^2 - 4\,xy - 5\,y^2 \\ -\,4\,ax \\ \hline -\,12\,ax^3 + 16\,ax^2y + 20\,axy^2. \end{array}$$
Ans.

2.
$$\begin{array}{r} 3\,x^2 - xy + y^2 \\ 3\,x \\ \hline 9\,x^3 - 3\,x^2y + 3\,xy^2. \end{array}$$
Ans.

3.
$$\begin{array}{l} x^2 + xy + y^2 \\ \qquad\qquad 3x \\ \hline 3x^3 + 3x^2y + 3xy^2. \quad \textbf{Ans.} \end{array}$$

4.
$$\begin{array}{l} ax + by + cz \\ \qquad\qquad axyz \\ \hline a^2x^2yz + abxy^2z + acxyz^2. \quad \textbf{Ans.} \end{array}$$

5.
$$\begin{array}{l} 3ax^3 - 5ay^2 - 7 \\ \qquad\qquad 9abx \\ \hline 27a^2bx^4 - 45a^2bxy^2 - 63abx. \quad \textbf{Ans.} \end{array}$$

6.
$$\begin{array}{l} 4mp - 6nq \\ \qquad -3mq \\ \hline -12m^2pq + 18mnq^2. \quad \textbf{Ans.} \end{array}$$

7.
$$\begin{array}{l} 5a^2y^3 - 7a^3y^2 - 7a^4y \\ \qquad\qquad\qquad 8ab \\ \hline 40a^3by^3 - 56a^4by^2 - 56a^5by. \quad \textbf{Ans.} \end{array}$$

§ 74.

1. $ap + mp - p^2 + bq - cq - br - cr.$

2. $mx - anx - my - any + anz - mz.$

3. $acx - acy - bdx + bdy + cdfx + cdfy.$

4. $= am[x - ab + ac] - bn[ax + bc + bd] =$
$amx - a^2bm + a^2cm - abnx - b^2cn - b^2dn.$

5. $= p[-am - an + bm - bn] - q[bm - bn - am - an] =$
$-amp - anp + bmp - bnp - bmq + bnq + amq + anq.$

6. $6qx - 3cnx + 10xy - 6cy - 2mz - 7nz.$

7. $= am[acm - bcm - 6hk + 12dh + 4n] =$
$a^2cm^2 - abcm^2 - 6ahkm + 12adhm + 4amn.$

8. $= 2pq[3a - 5b - 6c - 2mpq + 3npq] =$
$6apq - 10bpq - 12cpq - 4mp^2q^2 + 6np^2q^2.$

9. $= bn[-7a - 7ab + 7bc - 3 + a + b] =$
$-7abn - 7ab^2n + 7b^2cn - 3bn + abn + b^2n =$
$-6abn - 7ab^2n + 7b^2cn - 3bn + b^2n.$

10. $= pq - pr + qr - pq + pr - qr = 0.$

§ 75.

2. $=(m+p-m-2p-3g)\,y=-(p+3g)\,y.$

4. $=(3a-2a-a+1)\,z+(-1+1)\,y=z.$

5. $(ab-bc+bd)\,xy$, or $b\,(a-c+d)\,xy.$

6. $(36ab-7)\,xy+(-24-a)\,x$, or $(36ab-7)\,xy-(24+a)x.$

7. $3x+(a-b-ma-nb)y$, or $3x+[a(1-m)-b(1+n)]y.$

8. $m\,(a-b)\,y+n\,(a-b)\,y.$

9. $(pr-2qr-4p^2+8qh)\,z.$

10. $(cn+bn)x+(-am-2bn)y$, or $n(c+b)x-(am+2bn)\,y.$

§ 76.

1. $=3x^2y^2-4x^2y+x^2y^3-2x^3y^2+x^2+2xy^3+5y^2x-7x-y^2-6$
$=(x^2+2x)\,y^3+(3x^2-2x^3+5x-1)\,y^2-4x^2y+x^2-7x-6.$

2. $=x^2y^4-x^2y^2+xy^3-xy+y^2-1=$
$x^2y^4+xy^3+(1-x^2)\,y^2-xy-1.$

3. $=x^2y^5-2x^2y^2+x^2y^4-2x^3y^3+xy^3-2xy+y^2-2$
$x^2y^5+x^2y^4+(x-2x^3)\,y^3+(1-2x^2)\,y^2-2xy-2.$

4. $=x^4y^5+3x^4y^3+x^3y^4+3x^3y^2+x^2y^3+3x^2y+xy^2+3x$
$=x^4y^5+x^3y^4+(3x^4+x^2)\,y^3+(3x^3+x)\,y^2+3x^2y+3x.$

§ 78.

1.
$$\begin{array}{l}
2a-bn^2-2bn^3\\
\qquad a+b\\
\hline
2a^2-abn^2-2abn^3\\
\qquad\qquad\qquad +2ab-b^2n^2-2b^2n^3\\
\hline
2a^2-abn^2-2abn^3+2ab-b^2n^2-2b^2n^3.
\end{array}$$
Ans.

2.
$$\begin{array}{l}
3m+2n-5abmn\\
\qquad a-b\\
\hline
3am+2an-5a^2bmn\\
\qquad\qquad\qquad -3bm-2bn+5ab^2mn\\
\hline
3am+2an-5a^2bmn-3bm-2bn+5ab^2mn.
\end{array}$$
Ans.

3.
$$\begin{array}{l}
2\,mn + pm + qn \\
\qquad\qquad m^2 - n^2 \\
\hline
2\,m^2n + pm^2 + m^2nq \\
\qquad\qquad\qquad\qquad - 2\,mn^2 - pmn^2 + qn^2 \\
\hline
2\,m^2n + pm^2 + m^2nq - 2\,mn^2 - pmn^2 - qn^2. \quad \textbf{Ans.}
\end{array}$$

4.
$$\begin{array}{l}
p^2 + q^2 + r^2 \\
pq + qr + rp \\
\hline
p^2q + pq^2 + pqr^2 \\
\qquad\qquad +p^2qr + q^2r + qr^2 \\
\qquad\qquad\qquad\qquad +p^2r + pq^2r + pr^2 \\
\hline
p^2q + pq^2 + pqr^2 + p^2qr + q^2r + qr^2 + p^2r + pq^2r + pr^2. \quad \textbf{Ans.}
\end{array}$$

5.
$$\begin{array}{l}
2\,a + 2\,b \\
2\,a - 3\,b \\
\hline
4\,a^2 + 4\,ab \\
\qquad - 6\,ab - 6\,b^2 \\
\hline
4\,a^2 - 2\,ab - 6\,b^2. \quad \textbf{Ans.}
\end{array}$$

6.
$$\begin{array}{l}
mx + ny \\
mx - ny \\
\hline
m^2x^2 + mnxy \\
\qquad - mnxy - n^2y^2 \\
\hline
m^2x^2 \qquad\qquad - n^2y^2. \quad \textbf{Ans.}
\end{array}$$

§ 79.

1.
$$\begin{array}{l}
3\,a^2 + 5\,a + 7 \\
2\,a^2 - 3\,a + 4 \\
\hline
6\,a^4 + 10\,a^3 + 14\,a^2 \\
\qquad - \;\; 9\,a^3 - 15\,a^2 - 21\,a \\
\qquad\qquad\qquad + 12\,a^2 + 20\,a + 28 \\
\hline
6\,a^4 + \quad a^3 + 11\,a^2 - \quad a + 28. \quad \textbf{Ans.}
\end{array}$$

2.
$$\begin{array}{l}
a^2 + ab + b^2 \\
\qquad a - b \\
\hline
a^3 + a^2b + ab^2 \\
\quad - a^2b - ab^2 - b^3 \\
\hline
a^3 \qquad\qquad\qquad - b^3. \quad \textbf{Ans.}
\end{array}$$

3.
$$\begin{array}{l}
a^3 + a^2 + ax^2 + x^3 \\
\qquad\qquad\qquad a - x \\
\hline
a^4 + a^3 + a^2x^2 + ax^3 \\
\qquad\qquad\qquad - ax^3 - a^3x - a^2x - x^4 \\
\hline
a^4 + a^3 + a^2x^2 \qquad\qquad - a^3x - a^2x - x^4. \quad \textbf{Ans.}
\end{array}$$

4.
$$\begin{array}{l}
a^3 - a^2 + a - 1 \\
a^2 - a + 1 \\
\hline
a^5 - a^4 + a^3 - a^2 \\
\qquad - a^4 + a^3 - a^2 + a \\
\qquad\qquad + a^3 - a^2 + a - 1 \\
\hline
a^5 - 2a^4 + 3a^3 - 3a^2 + 2a - 1.\quad \text{Ans.}
\end{array}$$

5.
$$\begin{array}{l}
x^4 + ax^3 + a^2x^2 + a^3x + a^4 \\
x - a \\
\hline
x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x \\
\qquad - ax^4 - a^2x^3 - a^3x^2 - a^4x - a^5 \\
\hline
x^5 \qquad\qquad\qquad\qquad\qquad - a^5.\quad \text{Ans.}
\end{array}$$

6.
$$\begin{array}{l}
a + bz + cz^2 + dz^3 \\
\qquad m - nz + pz^2 \\
\hline
am + bmz + cmz^2 + dmz^3 \\
\qquad\qquad - anz - bnz^2 - cnz^3 - dnz^4 \\
\qquad\qquad\qquad\qquad + apz^2 + bpz^3 + cpz^4 + dpz^5 \\
\hline
am + bmz + cmz^2 + dmz^3 - anz - bnz^2 - cnz^3 - dnz^4 + apz^2 + \\
\qquad bpz^3 + cpz^4 + dpz^5 = \\
am + dpz^5 + (cp - dn)z^4 + (dm - cn + bp)z^3 + (cm - bn + \\
\qquad ap)z^2 + (bm - an)z.\quad \text{Ans.}
\end{array}$$

7.
$$\begin{array}{l}
3a^2 + 5a + 7 \\
2a^2 + 3a - 4 \\
\hline
6a^4 + 10a^3 + 14a^2 \\
\qquad + 9a^3 + 15a^2 + 21a \\
\qquad\qquad - 12a^2 - 20a - 28 \\
\hline
6a^4 + 19a^3 + 17a^2 + a - 28.\quad \text{Ans.}
\end{array}$$

8.
$$\begin{array}{l}
a^2 - ab + b^2 \\
a + b \\
\hline
a^3 - a^2b + ab^2 \\
\qquad + a^2b - ab^2 + b^3 \\
\hline
a^3 \qquad\qquad\qquad + b^3.\quad \text{Ans.}
\end{array}$$

9.
$$\begin{array}{l}
a^3 + a^2x + ax^2 + x^3 \\
a - x \\
\hline
a^4 + a^3x + a^2x^2 + ax^3 \\
\qquad - a^3x - a^2x^2 - ax^3 - x^4 \\
\hline
a^4 \qquad\qquad\qquad\qquad - x^4.\quad \text{Ans.}
\end{array}$$

10.
$$\begin{array}{l}
a^3 - a^2 + a - 1\\
a^2 + a - 1\\
\hline
a^5 - a^4 + a^3 - a^2\\
\quad + a^4 - a^3 + a^2 - a\\
\qquad\quad - a^3 + a^2 - a + 1\\
\hline
a^5 \qquad - a^3 + a^2 - 2a + 1.
\end{array}$$
Ans.

11.
$$\begin{array}{l}
x^4 + ax^3 + a^2x^2 + a^3x + a^4\\
x + a\\
\hline
x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x\\
\quad + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5\\
\hline
x^5 + 2ax^4 + 2a^2x^3 + 2a^3x^2 + 2a^4x + a^5.
\end{array}$$
Ans.

12.
$$\begin{array}{l}
a + bz + cz^2 + dz^3\\
m + nz - pz^2\\
\hline
am + bmz + cmz^2 + dmz^3\\
\qquad\qquad + anz + bnz^2 + cnz^3 + dnz^4\\
\qquad\qquad\qquad\qquad - apz^2 - bpz^3 - cpz^4 - dpz^5\\
\hline
am + bmz + cmz^2 + dmz^3 + anz + bnz^2 + cnz^3 + dnz^4 - apz^2 -\\
\qquad\qquad\qquad\qquad bpz^3 - cpz^4 - dpz^5 =\\
am \quad dpz^5 + (dn - cp)z^4 + (dm + cn - bp)z^3 + (cm + bn -\\
\qquad\qquad\qquad\qquad ap)z^2 + (bm + an)z.
\end{array}$$
Ans.

13.
$$\begin{array}{l}
a + bx\\
m + nx\\
\hline
am + bmx\\
\qquad\quad + anx + bnx^2\\
\hline
am + bmx + anx + bnx^2 = am + (bm + an)x + bnx^2.
\end{array}$$
Ans.

14.
$$\begin{array}{l}
a + bx + cx^2\\
m + nx + px^2\\
\hline
am + bmx + cmx^2\\
\qquad\qquad + anx + bnx^2 + cnx^3\\
\qquad\qquad\qquad\qquad + apx^2 + bpx^3 + cpx^4\\
\hline
am + bmx + cmx^2 + anx + bnx^2 + cnx^3 + apx^2 + bpx^3 + cpx^4 =\\
am + cpx^4 + (cn + bp)x^3 + (cm + ap + bn)x^2 +\\
\qquad\qquad\qquad\qquad (bm + an)x.
\end{array}$$
Ans.

15.
$$\begin{array}{l}
y^3 - 3y + 2\\
y^2 - 2\\
\hline
y^5 - 3y^3 + 2y^2\\
\quad - 2y^3 \qquad + 6y - 4\\
\hline
y^5 - 5y^3 + 2y^2 + 6y - 4.
\end{array}$$
Ans.

16.
$$
\begin{array}{l}
y^3+y^2+y+1 \\
\underline{y^2+y+1\qquad\qquad} \\
y^5+y^4+y^3+y^2 \\
\quad +y^4+y^3+y^2+y \\
\qquad\quad +y^3+y^2+y+1 \\
\hline
y^5+2y^4+3y^3+3y^2+2y+1. \quad \text{Ans.}
\end{array}
$$

17.
$$
\begin{array}{l}
y^3-2y^2+3y-4 \\
\underline{y^3+2y^2+3y+4\qquad} \\
y^6-2y^5+3y^4-4y^3 \\
\quad +2y^5-4y^4+6y^3-8y^2 \\
\qquad\quad +3y^4-6y^3+9y^2-12y \\
\qquad\qquad\quad +4y^3-8y^2+12y-16 \\
\hline
y^6 \qquad +2y^4 \qquad -7y^2 \qquad -16. \quad \text{Ans.}
\end{array}
$$

18.
$$
\begin{array}{l}
3a^{2m}x-3a^2y+2a^{2n} \\
\underline{\;a^m \quad -a^n\qquad\qquad\qquad} \\
3a^{3m}x-3a^{2+m}y+2a^{2n+m} \\
\qquad\qquad\qquad -3a^{2m+n}x+3a^{2+n}y-2a^{3n} \\
\hline
3a^{3m}x-3a^{2+m}y+2a^{2n+m}-3a^{2m+n}x+3a^{2+n}y-2a^{3n}. \quad \text{Ans.}
\end{array}
$$

19.
$$
\begin{array}{l}
a^2+6ab+\frac{1}{3}b \\
\underline{a-\frac{1}{3}b\qquad\qquad} \\
a^3+6a^2b+\frac{1}{3}ab \\
\quad -\frac{1}{3}a^2b-2ab^2-\frac{1}{9}b^2 \\
\hline
a^3+\frac{17}{3}a^2b+\frac{1}{3}ab-2ab^2-\frac{1}{9}b^2. \quad \text{Ans.}
\end{array}
$$

20.
$$
\begin{array}{l}
(a+b)+(a-b)=2a \\
(a+b)-(a-b)=\underline{2b} \\
\qquad\qquad\qquad\quad 2a\times 2b=4ab. \quad \text{Ans.}
\end{array}
$$

21.
$$
\begin{array}{l}
a^2-b^2+(a-b) \\
\underline{a^2+b^2+(a+b)} \\
a^4-a^2b^2+a^2(a-b) \\
\quad +a^2b^2 \qquad\qquad -b^4+b^2(a-b) \\
\qquad\qquad\qquad\qquad +a^2(a+b)-b^2(a+b)+(a^2-b^2) \\
\hline
a^4 \qquad +a^2(a-b)-b^4+b^2(a-b)+a^2(a+b)-b^2 \\
\qquad\qquad\qquad\qquad (a+b)+(a^2-b^2)= \\
a^4+a^3-a^2b-b^4+ab^2-b^3+a^3+a^2b-ab^2-b^3+a^2-b^2= \\
a^4+2a^3+a^2-b^2-2b^3-b^4. \quad \text{Ans.}
\end{array}
$$

22.
$$
\begin{array}{llll}
a+b+c \\
\underline{a-b+c} \\
a^2+ab+ac \\
\quad\;\; -ab & -b^2-bc \\
\qquad\quad +ac & +bc+c^2 \\
\hline
a^2 \quad +2ac-b^2 & \quad +c^2. & \textbf{Ans.}
\end{array}
$$

23.
$$
\begin{array}{rcl}
a^2+b^2-(3a^2+b^2) &=& -2a^2 \\
2a+2b-2(a-b) &=& \underline{\quad 4b \quad} \\
&& -2a^2 \times 4b = -8a^2b. \quad \textbf{Ans.}
\end{array}
$$

24.
$$
\begin{array}{l}
2(a-b)+x-y = 2a-2b+x-y \\
\underline{a+b-(x+y) \;=\; a+b-x-y} \\
2a^2-2ab+ax-ay \\
\qquad +2ab \qquad\qquad -2b^2+bx-by \\
\qquad\qquad -2ax \qquad\qquad +2bx \qquad -x^2+xy \\
\qquad\qquad\qquad -2ay \qquad\qquad +2by \qquad -xy+y^2 \\
\hline
2a^2 \quad -ax-3ay-2b^2+3bx+by-x^2 \quad +y^2= \\
2a^2-x^2-2b^2+(3b-a)x+(b-3a)y+y^2. \quad \textbf{Ans.}
\end{array}
$$

25.
$$
\begin{array}{l}
ax^m+bx^n-abx \\
\underline{ax^2+bx^3} \\
a^2x^{m+2}+abx^{n+2}-a^2bx^3 \\
\qquad\qquad\qquad\qquad +abx^{m+3}+b^2x^{n+3}-ab^2x^4 \\
\hline
a^2x^{m+2}+abx^{n+2}-a^2bx^3+abx^{m+3}+b^2x^{n+3}-ab^2x^4. \quad \textbf{Ans.}
\end{array}
$$

26.
$$
\begin{array}{l}
a^m-b^n \\
\underline{a^m+b^n} \\
a^{2m}-a^mb^n \\
\qquad +a^mb^n-b^{2n} \\
\hline
a^{2m} \qquad\quad -b^{2n}. \quad \textbf{Ans.}
\end{array}
$$

27.
$$
\begin{array}{r}
-15x^2+3xy^2-12y^3 \\
\underline{-\;5xy} \\
75x^3y-15x^2y^3+60xy^4. \quad \textbf{Ans.}
\end{array}
$$

28.
$$
\begin{array}{l}
\frac{2}{3}x^2+3ax-\frac{7}{5}a^2 \\
\underline{2x^3-ax-\frac{1}{4}a^2} \\
\frac{4}{3}x^5+6ax^4-\frac{14}{5}a^2x^3 \\
\qquad\qquad\qquad -\frac{2}{3}ax^3-3a^2x^2+\frac{7}{5}a^3x \\
\qquad\qquad\qquad\qquad\qquad -\frac{1}{6}a^2x^2-\frac{3}{4}a^3x+\frac{7}{20}a^4 \\
\hline
\frac{4}{3}x^5+6ax^4-\frac{14}{5}a^2x^3-\frac{2}{3}ax^3-\frac{19}{6}a^2x^2+\frac{13}{20}a^3x+\frac{7}{20}a^4. \quad \textbf{Ans.}
\end{array}
$$

§ 80.

1. $m^2 + 4mn + 4n^2$.
2. $m^2 - 4mn + 4n^2$.
3. $9a^2 - 12ab + 4b^2$.
4. $16x^2 - 40xy + 25y^2$.
5. $4x^2 - y^2$.
6. $9x^2 - 1$.
7. $16x^4 - 1$.
8. $25x^6 - 9$.

§ 82.

1. $a^2 + 2ab + b^2$.
2. $-x^2 + 2xy - y^2$.
3. $-x^2 - 2xy - y^2$.
4. $-4a^2 + 12ab - 9b^2$.
5. $-9b^2 + 12ab - 4a^2$.
6. $-a^2m^2 + 2ambn - b^2n^2$.
7. $-4 + 4xy - x^2y^2$.

§ 85.

2. $3a^2$
6. $2a^2m$.
7. $2a^2m^2$.
8. $6yz^2$.
9. $4x^2z$.
10. $5b^4$.

§ 86.

9. $-4ax^{m-2}$.
10. $-2b^{s-r}p^{t-q}$.
11. $3b^{m-n}$.
12. $4(a-b)c^2$.
13. $-6(x-y)^{m-n}$.
14. $-4a^{s-t}$.
15. $-5b^{m-n}(a-b)^{n-s}$.
16. $6(m+n)^{p-q}$.
17. $16(a+b)^{n-1}(x-y)^{m-1}$.

§ 87.

2. $6mn \,|\, \dfrac{6m^2n - 12m^3n^2 - 18mn^5}{m - 2m^2n - 3n^4}$. **Ans**

3. $4a^2b^3 \,|\, \dfrac{8a^2b^6 - 16a^3b^4 + 8a^4b^3}{2b^3 - 4ab + 2a^2}$. **Ans.**

4. $$\begin{array}{r|l} -4xy & 4xy^5 - 8x^3y^3 + 4x^5y \\ \hline & -y^4 + 2x^2y^2 - x^4. \end{array}$$ Ans.

5. $$\begin{array}{r|l} -12abx & 12abx - 24abx^2 \\ \hline & -1 + 2x. \end{array}$$ Ans.

6. $$\begin{array}{r|l} -7amx^n & 21am^2x^m - 14a^3m^4x^n + 28a^5m^2x^p \\ \hline & -3mx^{m-n} + 2a^2m^3 - 4a^4mx^{p-n}. \end{array}$$ Ans.

7. $$\begin{array}{r|l} 24ax & 72a^3x + 24ax + 48ax^2 \\ \hline & 3a^2 + 1 + 2x. \end{array}$$ Ans.

8. $$ab - ac + bc - ab + ac - bc + abc = abc$$
$$\begin{array}{r|l} abc & abc \\ \hline & 1. \end{array}$$ Ans.

9. $$\begin{array}{r|l} 9(a-b) & 27(a-b)^5 - 18(a-b)^3 + 9(a-b)^2 \\ \hline & 3(a-b)^4 - 2(a-b)^2 + (a-b). \end{array}$$ Ans.

10. $$\begin{array}{r|l} a^n(a-b)^n & a^m(a-b)^n - a^n(a-b)^m \\ \hline & a^{m-n} - (a-b)^{m-n}. \end{array}$$ Ans.

11. $$\begin{array}{r|l} (a+b)(a-b) & (a+b)^p(a-b)^q + (a+b)^q(a-b)^p \\ \hline & (a+b)^{p-1}(a-b)^{q-1} + (a+b)^{q-1}(a-b)^{p-1}. \end{array}$$ Ans.

12. $$\begin{array}{r|l} 5(x+y)(x-y) & 10(x+y)^m(x-y)^n - 5(x+y)^p(x-y)^q \\ \hline & 2(x+y)^{m-1}(x-y)^{n-1} - (x+y)^{p-1}(x-y)^{q-1}. \end{array}$$ Ans.

13. $$(a+b)(a-b) = a^2 - b^2$$
$$\begin{array}{r|l} a^2 - b^2 & a^2 - b^2 \\ \hline & 1. \end{array}$$ Ans.

§ 89.

1. $ax(x+a)$.

2. $abcy(a^2b + ac^2 + b^2c)$.

3. $a^n b^n (a^n + b^n)$.

4. $a^n x^n (a^{2n} - a^n x^n + x^{2n})$.

5. $a^n b^n c^n (b^n c^{2n} + a^n b^{2n} + a^{2n} c^n)$.

§ 90.

2. $(y^2 + 4x^2)(y + 2x)(y - 2x)$.

4. $(x - 3)^2$

5. $(2ax + 3by)(2ax - 3by)$.

6. $(4a^2x^2 + 1)(2ax + 1)(2ax - 1)$.

7. $(3x - 2y)^2$.

8. $(ax + y)^2$.

9. $(2ax + by)^2$.

10. $(a^2 + 2b)^2$.

11. $= (x^2 - y^2)^2 = [(x + y)(x - y)]^2$.

12. $(x^2 - 2y^2)^2$.

13. $(a^2 - 2b^2)^2$.

14. $(a^2 + ab)(a^2 - ab)$.

15. $(a^n - 1)^2$.

16. $(x^n - 2a)^2$.

17. $(1 + y^2)(1 + y)(1 - y)$.

19. $= a(a^2 - 4ab + 4b^2) = a(a - 2b)^2$.

20. $(a^m + b^{2n})(a^m - b^{2n})$.

21. $(5x^2 - 4xy)^2$

22. $(2x^2y^2 + 3xy)(2x^2y^2 - 3xy)$.

23. $(2x^2y^2 - 3x)^2$.

24. $(x^4 + xy^2)(x^4 - xy^2)$.

25. $(x^{2m} - y^n)^2$.

26. $(x^{2m} - 1)^2$.

27. $(x + \frac{1}{2})^2$.

28. $(x^m + \frac{1}{2})^2$.

§ 91.

1. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.

2. $(a^2 + 4b^2)^2$.

3. $(a^2 + 3ax + 9x^2)(a^2 - 3ax + 9x^2)$.

4. $(a^{2n} + a^n b^n + b^{2n})(a^{2n} - a^n b^n + b^{2n})$.

5. $a^2x + 2abx + 4b^2x)(a^2x - 2abx + 4b^2x)$.

6. $a^2(a^2 + 4b^2)^2$.

7. $= x^n(x^{4n} + x^{2n}y^{2n} + y^{4n}) =$
$x^n(x^{2n} + x^n y^n + y^{2n})(x^{2n} - x^n y^n + y^{2n})$.

9. $= a^2 - (4b^2 - 4bc + c^2) = a^2 - (2b - c)^2 =$
$[a + (2b - c)][a - (2b - c)] = (a + 2b - c)(a - 2b + c)$.

10. $= a(a^2 - 4b^2 + 4bc - c^2) = a[a^2 - (4b^2 - 4bc + c^2) =$
$a[a^2 - (2b - c)^2 = a[a - (2b - c)][a + (2b - c)] =$
$a(a - 2b + c)(a + 2b - c)$.

§ 94.

2. $= (5 - 2)(5^2 + 5.2 + 2^2)$.
Proof. $5^3 - 2^3 = 125 - 8 = 117$.
$(5-2)(5^2 + 5.2 + 2^2) = 3(25 + 10 + 4) = 3 \times 39 = 117$.

3. $= (5^2 + 2^2)(5 + 2)(5 - 2)$.
Proof. $5^4 - 2^4 = 625 - 16 = 609$.
$(5^2 + 2^2)(5 + 2)(5 - 2) = (25 + 4)\, 7 \times 3 = 29 \times 7 \times 3 = 609$.

4. $(5 - 2)(5^4 + 5^3.2 + 5^2.2^2 + 5.2^3 + 2^4)$.
Proof. $5^5 - 2^5 = 3125 - 32 = 3093$.
$(5 - 2)(5^4 + 5^3.2 + 5^2.2^2 + 5.2^3 + 2^4) =$
$3(625 + 250 + 100 + 40 + 16) = 3 \times 1031 = 3093$.

5. $[(5 - 2)(5^2 + 5.2 + 2^2)][(5 + 2)(5^2 - 5.2 + 2^2)]$.
Proof. $5^6 - 2^6 = 15625 - 64 = 15561$.
$[(5^2 + 5.2 + 2^2)(5 - 2)][(5 + 2)(5^2 - 5.2 + 2^2)] =$
$(25 + 10 + 4)\, 3 \times 7\, (25 - 10 + 4) = (39 \times 3) \times (7 \times 19) =$
$117 \times 133 = 15561$.

6. $(7+2)(7^2-7.2+2^2)$.
Proof. $7^3+2^3=343+8=351$.
$(7+2)(7^2-7.2+2^2)=9(49-14+4)=9\times39=351$.

7. $(7-2)(7^2+7.2.+2^2)$.
Proof. $7^3-2^3=343-8=335$.
$(7-2)(7^2+7.2+2^2)=5(49+14+4)=5\times67=335$.

8. $(7^2+2^2)(7+2)(7-2)$.
Proof. $7^4-2^4=2401-16=2385$.
$(7^2+2^2)(7+2)(7-2)=(49+4)9\times5=53\times9\times5=2385$.

9. $(x+a)(x-a)$.

10. $(x-a)(x^2+xa+a^2)$.

11. $(x^2+a^2)(x+a)(x-a)$.

12. $(x-a)(x^4+x^3a+x^2a^2+xa^3+a^4)$.

13. $(x+a)(x^2-xa+a^2)$.

14. $(x+a)(x^4-x^3a+x^2a^2-xa^3+a^4)$.

15. $(a-2b)(a^2+2ab+4b^2)$.

16. $(2a-3b)(4a^2+6ab+9b^2)$.

17. $(4a^2+b^2)(2a+b)(2a-b)$.

18. $(x+2y)(x^2-2xy+4y^2)$.

19. $(x^2+4y^2)(x+2y)(x-2y)$.

20. $(2a+3b)(4a^2-6ab+9b^2)$.

21. $(x-2a)(x^2+2ax+4a^2)(x+2a)(x^2-2ax+4a^2)$.

§ 95.

1. xyz.

2. $a^2b^2c^2d^2$.

3. $abcd$.

4. $a^2b^3c^4$.

5. $=(x+y)(x-y),\ x+y,\ x-y,$
$\therefore$ L.C.M. $=(x+y)(x-y)=x^2-y^2$.

6. $=(x^2+2)(x^2-2),\ (x-2)^2,\ (x+2)^2,$
$\therefore$ L.C.M. $=(x^2+2)(x^2-2)(x-2)^2(x+2)^2$.

7. $= (4ax - 2m)(4ax + 2m),\ 2ax + m,\ 2ax - m,$
$\therefore$ L.C.M. $= (4ax - 2m)(4ax + 2m).$

8. $= (x + 1)(x - 1),\ (x^2 + 1),\ (x - 1)^2,\ (x + 1)^2,$
$\therefore$ L.C.M. $= (x^2 + 1)(x - 1)^2(x + 1)^2.$

9. $4ab(b + c)(a - c).$

10. $2(a - b)^2(a + b)^2.$

11. $= 3(x + y),\ 3(x - y),\ 3(x + y)(x^2 - xy + y^2),$
$\therefore$ L.C.M. $= 3(x + y)(x - y)(x^2 - xy + y^2).$

12. $= a - b,\ (a + b)(a - b),\ (a - b)(a^2 + ab + b^2),\ (a^2 + b^2)(a + b)(a - b),$
$\therefore$ L.C.M. $= (a - b)(a + b)(a^2 + ab + b^2)(a^2 + b^2).$

13. $(x + y)(x - y)(a + b)(a - b) = (x^2 - y^2)(a^2 - b^2).$

14. $= (x^2 + a^2)(x + a)(x - a),\ (x + a)(x^2 - ax + x^2),\ (x - a)(x + a),\ (x + a),$
$\therefore$ L.C.M. $= (x^2 + a^2)(x + a)(x - a)(x^2 - ax + x^2).$

15. $= (x + 2a)(x^2 - 2ax + 4ax^2)(x - 2a)(x^2 + 2ax + 4a^2),$
$(x^2 + 4a^2)(x + 2a)(x - 2a),\ (x - 2a)(x + 2a),$
$\therefore$ L.C.M. $= (x + 2a)(x - 2a)(x^2 + 4a^2)(x^2 - 2ax + 4a^2)(x^2 + 2ax + 4a^2).$

16. $= a + b,\ (a + b)^2,\ (a^2 + b^2)(a + b)(a - b),$
$\therefore$ L.C.M. $(a + b)^2(a^2 + b^2)(a - b).$

§ 96.

1.
$$\begin{array}{l|l}
x^2 - 2x - 1 & x + 1 \\ \cline{2-2}
x^2 + x & x - 3. \quad \text{Ans.} \\ \cline{1-1}
\quad - 3x - 1 & \\
\quad - 3x - 3 & \\ \cline{1-1}
\qquad\qquad 2 & \\
\end{array}$$
$\frac{2}{x + 1}$. Rem.

2.
$$\begin{array}{l|l}
x^3 + 2x^2 - 2x - 1 & x - 1 \\ \cline{2-2}
x^3 - x^2 & x^2 + 3x + 1. \quad \text{Ans.} \\ \cline{1-1}
\quad + 3x^2 - 2x & \\
\quad + 3x^2 - 3x & \\ \cline{1-1}
\qquad\qquad + x - 1 & \\
\qquad\qquad\ \ x - 1 & \\ \cline{1-1}
\end{array}$$

3.
$$\begin{array}{l|l} x^3 - 3x^2 + 2x - 1 & x^2 - x \\ \hline x^3 - x^2 & x - 2. \quad \textbf{Ans.} \\ \hline -2x^2 + 2x & \\ -2x^2 + 2x & \\ \hline -1 & \end{array}$$

$$\frac{-1}{x^2 - x}. \quad \textbf{Rem.}$$

4.
$$\begin{array}{l|l} 2x^4 - 2x^3 + x^2 - x - 5 & x^2 - x - 1 \\ \hline 2x^4 - 2x^3 - 2x^2 & 2x^2 + 3. \quad \textbf{Ans.} \\ \hline 3x^2 - x - 5 & \\ 3x^2 - 3x - 3 & \\ \hline 2x - 2 & \end{array}$$

$$\frac{2x - 2}{x^2 - x - 1} = \frac{2(x - 1)}{x^2 - x - 1}. \quad \textbf{Rem.}$$

7.
$$\begin{array}{l|l} a^3 - 2a + 1 & a - 1 \\ \hline a^3 - a^2 & a^2 + a - 1. \quad \textbf{Ans.} \\ \hline +a^2 - 2a + 1 & \\ a^2 - a & \\ \hline -a + 1 & \\ -a + 1 & \\ \hline \end{array}$$

8.
$$\begin{array}{l|l} x^2 + 1 & x + 1 \\ \hline x^2 + x & x - 1. \quad \textbf{Ans.} \\ \hline -x + 1 & \\ -x - 1 & \\ \hline 2 & \end{array}$$

$$\frac{2}{x + 1}. \quad \textbf{Rem}$$

9.
$$\begin{array}{l|l} 8a^3 + 125 & 2a + 5 \\ \hline 8a^3 + 20a^2 & 4a^2 - 10a + 25. \quad \textbf{Ans.} \\ \hline -20a^2 + 125 & \\ -20a^2 - 50a & \\ \hline +50a + 125 & \\ +50a + 125 & \\ \hline \end{array}$$

10.
$$\begin{array}{l|l} a^5 + 1 & a + 1 \\ \hline a^5 + a^4 & a^4 - a^3 + a^2 - a + 1. \quad \textbf{Ans.} \\ \hline -a^4 + 1 & \\ -a^4 - a^3 & \\ \hline +a^3 + 1 & \\ +a^3 + a^2 & \\ \hline -a^2 + 1 & \\ -a^2 - a & \\ \hline +a + 1 & \end{array}$$

11.
$$\begin{array}{l|l}
a^4 + 2a^2 + 9 & a^2 + 2a + 3 \\
\cline{2-2}
a^4 + 2a^3 + 3a^2 & a^2 - 2a + 3. \quad \textbf{Ans.} \\
\cline{1-1}
\quad - 2a^3 - a^2 + 9 & \\
\quad - 2a^3 - 4a^2 - 6a & \\
\cline{1-1}
\qquad + 3a^2 + 6a + 9 & \\
\qquad + 3a^2 + 6a + 9 &
\end{array}$$

12.
$$\begin{array}{l|l}
a^6 - 1 & a^3 + 2a^2 + 2a + 1 \\
\cline{2-2}
a^6 + 2a^5 + 2a^4 + a^3 & a^3 - 2a^2 + 2a - 1. \quad \textbf{Ans.} \\
\cline{1-1}
\quad - 2a^5 - 2a^4 - a^3 - 1 & \\
\quad - 2a^5 - 4a^4 - 4a^3 - 2a^2 & \\
\cline{1-1}
\qquad + 2a^4 + 3a^3 + 2a^2 - 1 & \\
\qquad + 2a^4 + 4a^3 + 4a^2 + 2a & \\
\cline{1-1}
\qquad\quad - a^3 - 2a^2 - 2a - 1 & \\
\qquad\quad - a^3 - 2a^2 - 2a - 1 &
\end{array}$$

13.
$$\begin{array}{l|l}
x^6 - 12x^4 + 36x^2 - 32 & x^2 - 2 \\
\cline{2-2}
x^6 - 2x^4 & x^4 - 10x^2 + 16. \quad \textbf{Ans.} \\
\cline{1-1}
\quad - 10x^4 + 36x^2 & \\
\quad - 10x^4 + 20x^2 & \\
\cline{1-1}
\qquad + 16x^2 - 32 & \\
\qquad + 16x^2 - 32 &
\end{array}$$

14.
$$\begin{array}{l}
x^3 - 2x + 1 \\
x^3 - 12x - 16 \\
\hline
x^6 - 2x^4 + x^2 \\
\quad - 12x^4 + 24x^2 - 12x \\
\qquad - 16x^3 + 32x - 16 \\
\hline
x^6 - 14x^4 - 15x^3 + 24x^2 + 20x - 16 \text{ prod. of div.}
\end{array}$$

$$\begin{array}{l|l}
x^6 - 14x^4 - 15x^3 + 24x^2 + 20x - 16 & x^2 - 16 \\
\cline{2-2}
x^6 - 16x^4 & x^4 + 2x^2 - 15x + 56. \quad \textbf{Ans.} \\
\cline{1-1}
\quad + 2x^4 - 15x^3 + 24x^2 & \\
\quad + 2x^4 - 32x^2 & \\
\cline{1-1}
\qquad - 15x^3 + 56x^2 + 20x & \\
\qquad - 15x^3 + 240x & \\
\cline{1-1}
\qquad\quad + 56x^2 - 220x - 16 & \\
\qquad\quad + 56x^2 - 896 & \\
\cline{1-1}
\end{array}$$

$$\frac{-220x + 880}{x^2 - 16} = \frac{-220(x-4)}{x^2-16} = \frac{-220}{x+4} \text{ rem.}$$

15.
$$\begin{array}{l|l}
1+3x+3x^2+x^3 & 1+x \\ \cline{2-2}
1+x & 1+2x+x^2. \quad \text{Ans.} \\ \cline{1-1}
\quad 2x+3x^2 & \\
\quad 2x+2x^2 & \\ \cline{1-1}
\qquad x^2+x^3 & \\
\qquad x^2+x^3 & \\ \cline{1-1}
\end{array}$$

16.
$$\begin{array}{l|l}
1-4x+4x^2-x^3 & 1-x \\ \cline{2-2}
1-x & 1-3x+x^2. \quad \text{Ans.} \\ \cline{1-1}
\quad -3x+4x^2 & \\
\quad -3x+3x^2 & \\ \cline{1-1}
\qquad x^2-x^3 & \\
\qquad x^2-x^3 & \\ \cline{1-1}
\end{array}$$

17.
$$\begin{array}{l|l}
15+2a-3a^2+a^3+2a^4-a^5 & 5+4a-a^2 \\ \cline{2-2}
15+12a-3a^2 & 3-2a+a^2. \\ \cline{1-1}
\quad -10a-3a^2+4a^3+2a^4 & \\
\quad -10a-8a^2+2a^4 & \\ \cline{1-1}
\qquad +5a^2+4a^3-a^5 & \\
\qquad 5a^2+4a^3-a^5 & \\ \cline{1-1}
\end{array}$$

18.
$$\begin{array}{l|l}
1-y^6 & 1+2y+2y^2+y^3 \\ \cline{2-2}
1+2y+2y^2+y^3 & 1-2y+2y^2-y^3. \quad \text{Ans.} \\ \cline{1-1}
\quad -2y-2y^2-y^3-y^6 & \\
\quad -2y-4y^2-4y^3-2y^4 & \\ \cline{1-1}
\qquad +2y^2+3y^3+2y^4-y^6 & \\
\qquad +2y^2+4y^3+4y^4+2y^5 & \\ \cline{1-1}
\qquad\quad -y^3-2y^4-2y^5-y^6 & \\
\qquad\quad -y^3-2y^4-2y^5-y^6 & \\ \cline{1-1}
\end{array}$$

19.
$$\begin{array}{l|l}
64-64x+16x^2-8x^3+4x^4-x^6 & -4+2x+x^2 \\ \cline{2-2}
64-32x-16x^2 & -16+8x-4x^2+2x^3-x^4. \quad \text{Ans.} \\ \cline{1-1}
\quad -32x+32x^2-8x^3 & \\
\quad -32x+16x^2+8x^3 & \\ \cline{1-1}
\qquad 16x^2-16x^3+4x^4 & \\
\qquad 16x^2-8x^3-4x^4 & \\ \cline{1-1}
\qquad\quad -8x^3+8x^4-x^6 & \\
\qquad\quad -8x^3+4x^4+2x^5 & \\ \cline{1-1}
\qquad\qquad +4x^4-2x^5-x^6 & \\
\qquad\qquad 4x^4-2x^5-x^6 & \\ \cline{1-1}
\end{array}$$

20.
$$\begin{array}{l|l}
64-16x^2+x^6 & 4-4x+x^2 \\ \cline{2-2}
64+16x^2-64x & 16+16x+8x^2+4x^3+2x^4. \quad \text{Ans.} \\ \cline{1-1}
\quad +64x-32x^2+x^6 & \\
\quad +64x-64x^2+16x^3 & \\ \cline{1-1}
\qquad +32x^2-16x^3+x^6 & \\
\qquad +32x^2-32x^3+8x^4 & \\ \cline{1-1}
\qquad\quad +16x^3-8x^4+x^6 & \\
\qquad\quad +16x^3-16x^4+4x^5 & \\ \cline{1-1}
\qquad\qquad +8x^4-4x^5+x^6 & \\
\qquad\qquad +8x^4-8x^5+2x^6 & \\ \cline{1-1}
\qquad\qquad\quad \dfrac{4x^5-x^6}{4-4x+x^2}\ \text{rem.} &
\end{array}$$

§ 97.

1.
$$\begin{array}{l|l}
x^3-ax^2+abx+acx-abc-bx^2-cx^2+bcx & x-b \\ \cline{2-2}
x^3 \qquad\qquad\qquad\qquad -bx^2 & x^2-ax+ac-cx= \\ \cline{1-1}
\quad -ax^2+abx & x^2-(a+c)x+ac. \quad \text{Ans.} \\
\quad -ax^2+abx & \\ \cline{1-1}
\qquad\qquad acx-abc & \\
\qquad\qquad acx-abc & \\ \cline{1-1}
\qquad\qquad\qquad -cx^2+bcx & \\
\qquad\qquad\qquad -cx^2+bcx & \\ \cline{1-1}
\end{array}$$

2.
$$\begin{array}{l|l}
x^3-ax^2+abx+acx-abc-bx^2-cx^2+bcx & x-c \\ \cline{2-2}
x^3 \qquad\qquad\qquad\qquad\qquad -cx^2 & x^2-ax+ab-bx= \\ \cline{1-1}
\quad -ax^2+abx+acx & x^2-(a+b)x+ab. \quad \text{Ans.} \\
\quad -ax^2 \qquad +acx & \\ \cline{1-1}
\qquad +abx \qquad -abc & \\
\qquad +abx \qquad -abc & \\ \cline{1-1}
\qquad\qquad\qquad -bx^2+bcx & \\
\qquad\qquad\qquad -bx^2+bcx & \\ \cline{1-1}
\end{array}$$

3.
$$\begin{array}{l|l}
a^3+b^3-c^3+3abc & a+b-c \\ \cline{2-2}
a^3+a^2b-a^2c & a^2-ab+ac+b^2+c^2+bc. \quad \text{Ans.} \\ \cline{1-1}
\quad -a^2b+a^2c+b^3-c^3+3abc & \\
\quad -a^2b-ab^2 \qquad\qquad +abc & \\ \cline{1-1}
\qquad +a^2c+ab^2+b^3-c^3+2abc & \\
\qquad +a^2c \qquad -ac^2 \qquad +abc & \\ \cline{1-1}
\qquad\quad +ab^2+ac^2+b^3+abc-c^3 & \\
\qquad\quad ab^2 \qquad +b^3-b^2c & \\ \cline{1-1}
\qquad\qquad ac^2+abc-c^3+b^2c & \\
\qquad\qquad ac^2 \qquad -c^3+bc^2 & \\ \cline{1-1}
\qquad\qquad\quad +abc+b^2c-bc^2 & \\
\qquad\qquad\quad +abc+b^2c-bc^2 & \\ \cline{1-1}
\end{array}$$

4.
$$\begin{array}{l|l}
a^3+b^3+3ab-1 & a+b-1 \\
\underline{a^3+a^2b-a^2} & \overline{a^2-ab+a+b^2+b+1}. \quad \text{Ans.} \\
\quad -a^2b+a^2+b^3+3ab-1 & \\
\quad \underline{-a^2b \qquad -ab^2+ab} & \\
\qquad +a^2+ab^2+2ab+b^3-1 & \\
\qquad \underline{a^2 \qquad +ab-a} & \\
\qquad\quad +ab^2+ab+a+b^3-1 & \\
\qquad\quad \underline{ab^2 \qquad +b^3-b^2} & \\
\qquad\qquad +ab+a+b^2-1 & \\
\qquad\qquad \underline{ab \qquad +b^2-b} & \\
\qquad\qquad\quad +a \qquad +b-1 & \\
\qquad\qquad\quad \underline{a \qquad +b-1} &
\end{array}$$

5.
$$\begin{array}{l|l}
a^2b^2+2abx^2-a^2x^2-b^2x^2 & ab+ax-bx \\
\underline{a^2b^2+\ a^2bx-ab^2x} & \overline{ab+bx-ax}. \quad \text{Ans.} \\
\quad +ab^2x-a^2bx+2abx^2-a^2x^2-b^2x^2 & \\
\quad \underline{+ab^2x \qquad +\ abx^2 \qquad -b^2x^2} & \\
\qquad -a^2bx+\ abx^2-a^2x^2 & \\
\qquad \underline{-a^2bx+\ abx^2-a^2x^2} &
\end{array}$$

6. $(a^2-bc)^3=(a^6-3a^4bc+3a^2b^2c^2-b^3c^3)+8b^3c^3=$
$$\begin{array}{l|l}
a^6-3a^4bc+3a^2b^2c^2+7b^3c^3 & a^2+bc \\
\underline{a^6+a^4bc} & \overline{a^4-4a^2bc+7b^2c^2}. \quad \text{Ans.} \\
\quad -4a^4bc+3a^2b^2c^2 & \\
\quad \underline{-4a^4bc-4a^2b^2c^2} & \\
\qquad +7a^2b^2c^2+7b^3c^3 & \\
\qquad \underline{+7a^2b^2c^2+7b^3c^3} &
\end{array}$$

7. $(ab+bc+ca)(a+b+c)-abc$
$$\begin{array}{l}
ab+bc+ca \\
\underline{a+b\ +c} \\
a^2b+abc+a^2c+ab^2+b^2c+bc^2+ac^2 \\
\quad +abc \\
\quad \underline{+abc} \\
a^2b+3abc+a^2c+ab^2+b^2c+bc^2+ac^2 \\
\quad \underline{-\ abc}
\end{array}$$

$$\begin{array}{l|l}
a^2b+2abc+a^2c+ab^2+b^2c+bc^2+ac^2 & a+b \\
\underline{a^2b \qquad\qquad +ab^2} & \overline{ab+ac+c^2+bc} \\
a^2c+2abc & =(a+c)(b+c). \text{ Ans.} \\
\underline{a^2c+\ abc} & \\
ac^2+\ abc \qquad\qquad +bc^2 & \\
\underline{ac^2 \qquad\qquad\qquad +bc^2} & \\
\qquad abc+b^2c & \\
\qquad abc+b^2c &
\end{array}$$

But exercises 7 and 8 can be most expeditiously done by factoring, thus:

$$\begin{aligned}
\text{In (7) Divid.} &= (a+b)(ab+bc+ca) \\
&\quad + c(ab+bc+ca) - abc \\
&= (a+b)(ab+bc+ca) + c[ab+(a+b)c] - abc \\
&= (a+b)(ab+bc+ca+c^2) + cab - abc \\
\therefore \text{Quot.} &= ab+bc+ca+c^2 = (a+c)(b+c). \text{ Ans.}
\end{aligned}$$

$$\begin{aligned}
\text{In (8) Divisor} &= a^2 - (b^2 + c^2 - 2bc) \\
&= a^2 - (b-c)^2 = (a+b-c)(a-b+c) \\
\therefore \text{Quot.} &= \frac{(a+b-c)(b+c-a)(c+a-b)}{(a+b-c)(a-b+c)} = \\
&\qquad b+c-a. \text{ Ans.}
\end{aligned}$$

9.
$$\begin{array}{l|l}
a^3+b^3+c^3-3abc & a+b+c \\ \hline
a^3+a^2b+a^2c & a^2-ab-ac+b^2-bc+c^2. \quad \text{Ans.} \\ \hline
-a^2b-a^2c+b^3+c^3-3abc & \\
-a^2b-ab^2-abc & \\ \hline
-a^2c+ab^2+b^3-2abc+c^3 & \\
-a^2c-abc-ac^2 & \\ \hline
+ab^2+b^3-abc+ac^2+c^3 & \\
+ab^2+b^3+b^2c & \\ \hline
-abc+ac^2-b^2c+c^3 & \\
-abc-b^2c-bc^2 & \\ \hline
+ac^2+bc^2+c^3 & \\
+ac^2+bc^2+c^3 & \\ \hline
\end{array}$$

10.
$$\begin{array}{l|l}
x^4+4a^4 & x^2-2ax+2a^2 \\ \hline
x^4-2ax^3+2a^2x^2 & x^2+2ax+2a^2. \quad \text{Ans.} \\ \hline
+2ax^3-2a^2x^2+4a^4 & \\
+2ax^3-4a^2x^2+4a^3x & \\ \hline
+2a^2x^2-4a^3x+4a^4 & \\
+2a^2x^2-4a^3x+4a^4 & \\ \hline
\end{array}$$

11. $a^2b + a^2x - b^2x + ab^2 + ax^2 - bx^2 + abx$,
arranging in order of the exponents of x, we have

$$\begin{array}{l|l}
ax^2-bx^2+a^2x-b^2x+abx+a^2b+ab^2 & x+a+b \\ \hline
ax^2+a^2x+abx & ax-bx+ab. \quad \text{Ans.} \\ \hline
-bx^2-b^2x+a^2b & \\
-bx^2-abx-b^2x & \\ \hline
+abx+a^2b+ab^2 & \\
+abx+a^2b+ab^2 & \\ \hline
\end{array}$$

12.
$$\begin{array}{l|l}
x^3 - ax^2 - b^2x + ab^2 & x^2 - ax + bx - ab \\
\underline{x^3 - ax^2 + bx^2 - abx} & \overline{\qquad\qquad x - b}. \quad \textbf{Ans.} \\
\quad - bx^2 - b^2x + abx + ab^2 & \\
\quad \underline{- bx^2 - b^2x + abx + ab^2} &
\end{array}$$

13.
$$\begin{array}{l|l}
12\,a^4x^9 - 14\,a^5x^6 + 6\,a^6x^3 - a^7 & 2\,a^2x^3 - a^3 \\
\underline{12\,a^4x^9 - \quad 6\,a^5x^6} & \overline{6\,a^2x^6 - 4\,a^3x^3 + a^4}. \quad \textbf{Ans.} \\
\qquad - \quad 8\,a^5x^6 + 6\,a^6x^3 & \\
\qquad \underline{- \quad 8\,a^5x^6 + 4\,a^6x^3} & \\
\qquad\qquad\qquad 2\,a^6x^3 - a^7 & \\
\qquad\qquad\qquad \underline{2\,a^6x^3 - a^7} &
\end{array}$$

§ 99.

1. Divide both num. and den. by a^2b^2p gives $\frac{a^2p}{b^2}$.

2. " " " am gives $\frac{1}{ax}$.

3. " " " $2pr^2$ gives $\frac{5q}{6pr^2}$.

4. " " " $3axy$ gives $\frac{4}{5axy}$.

5. " " " $36(a-x)$ gives $2\frac{b-c}{a-x}$.

6. " " " $4(m-n)$ gives $\frac{5(a+x)}{6(a^2-2ax+x^2)}$.

7. " " " $a-b$ gives $\frac{y}{x}$.

8. $= \frac{(ay-by)(ay+by)}{(ay-by)}$ gives $ay+by$.

9. $= \dfrac{(a-b)(a+b)}{(a-b)(a-b)}$, divide both num. and den. by $a-b$ gives $\dfrac{a+b}{a-b}$.

10. $= \dfrac{(a+2x)(a+2x)}{(a+2x)(a-2x)}$, divide both num. and den. by $a+2x$ gives $\dfrac{a+2x}{a-2x}$.

11. $= \dfrac{(x+y)(x^2-xy+y^2)}{a(x+y)}$, divide both num. and den. by $x+y$ gives $\dfrac{x^2-xy+y^2}{a}$.

12. $= \dfrac{(a+2b)(a^2-2ab+4b^2)}{y(a+2b)}$, div. both num. and den. by $(a+2b)$ gives $\dfrac{a^2-2ab+4b^2}{y}$.

13. $= \dfrac{(a^2+b^2)(a^2-b^2)}{a^2-b^2}$, divide both num. and den. by a^2-b^2 gives a^2+b^2.

14. $= \dfrac{a^2+ab+b^2}{(a^2+ab+b^2)(a^2-ab+b^2)}$, divide both num. and den. by a^2+ab+b^2 gives $\dfrac{1}{a^2-ab+b^2}$.

15. $= \dfrac{(x+y)(x-y)}{(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)}$, divide both num. and den. by $x-y$ gives $\dfrac{x+y}{x^4+x^3y+x^2y^2+xy^3+y^4}$.

16. $= \dfrac{(x^n-y^n)(x^n+y^n)}{x^n+y^n}$, divide both num. and den. by x^n+y^n gives x^n-y^n.

17. $= \dfrac{ax(m-n)}{by(m-n)}$, divide both num. and den. by $m-n$ gives $\dfrac{ax}{by}$.

18. $= \frac{x(m-n)}{(a+b)(m-n)}$, divide both num. and den. by $m-n$ gives $\frac{x}{a+b}$.

§ 100.

1. $\frac{x-y}{a} = \frac{y-x}{-a} = -\frac{x-y}{-a} = -\frac{y-x}{a}$.

2. $\frac{x-y}{a-b} = \frac{y-x}{b-a} = -\frac{x-y}{b-a} = -\frac{y-x}{a-b}$.

3. $\frac{m}{p-q} = \frac{-m}{q-p} = -\frac{m}{q-p} = -\frac{-m}{p-q}$.

4. $\frac{a}{a-b+c} = \frac{-a}{b-a-c} = -\frac{a}{b-a-c} = -\frac{-a}{a-b+c}$.

5. $-\frac{m-n}{p+q-r} = -\frac{n-m}{r-p-q} = \frac{n-m}{p+q-r} = \frac{m-n}{r-p-q}$.

6. $\frac{a+m-x}{a-m+x} = \frac{x-a-m}{m-a-x} = -\frac{a+m-x}{m-a-x} = -\frac{x-a-m}{a-m+x}$.

7. $-\frac{x-b}{y-c}$.

8. $\frac{x-m}{y-n}$.

9. $-\frac{a+x-b}{x-a-b}$.

10. $-\frac{x-a}{x-b}$.

11. $\frac{x-a+b}{x-b}$.

12. $-\frac{x-a-b}{a-b+y}$.

§ 101.

1. $= p\frac{qx}{mn} = pq\frac{x}{mn} = pqx\frac{1}{mn}$.

2. $= a\frac{b}{c} = ab\frac{1}{c}$.

3. $= a\frac{bc}{a+b} = ab\frac{c}{a+b} = abc\frac{1}{a+b}$.

4. $= (x^2 - y^2)\dfrac{1}{a-b} = (x+y)\dfrac{x-y}{a-b} = (x-y)\dfrac{x+y}{a-b} =$
$(x+y)(x-y)\dfrac{1}{a-b}.$

5. $= (a^4 - b^4)\dfrac{1}{x} = (a^2+b^2)\dfrac{a^2-b^2}{x} = (a^2+b^2)\dfrac{(a-b)(a+b)}{x}$
$= (a^2+b^2)(a+b)\dfrac{a-b}{x} = (a^2+b^2)(a+b)(a-b)\dfrac{1}{x}.$

6. $= (x^4 - 16a^4)\dfrac{1}{x+2a} = (x^2+4a^2)\dfrac{(x^2-4a^2)}{x+2a} =$
$(x^2+4a^2)\dfrac{(x+2a)(x-2a)}{x+2a} =$
$(x^2+4a^2)(x+2a)\dfrac{x-2a}{x+2a} =$
$(x^2+4a^2)(x+2a)(x-2a)\dfrac{1}{x+2a} =$
$(x^2+4a^2)(x-2a).$

§ 102.

1. $\dfrac{ab}{b}.$

2. $\dfrac{a^2x^2}{ax}.$

3. $\dfrac{a^2b^{1+n}}{ab^n}.$

4. $\dfrac{m(x-y)}{n(x-y)}.$

5. $-\dfrac{x}{x}.$

6. $\dfrac{m(n-p)(a-b)}{a^2-b^2}$ or $\dfrac{m(n-p)(a-b)}{(a+b)(a-b)}.$

7. $\dfrac{(x+y)(x+y)}{x^2-y^2} = \dfrac{x^2+2xy+y^2}{x^2-y^2 \text{ or } (x+y)(x-y)}.$

8. $\dfrac{(x^2+1)(x+1)}{x^2+2x+1}$ or $\dfrac{(x^2+1)(x+1)}{(x+1)(x+1)}.$

9. $\dfrac{(a+1)(a^3+a^2+a+1)}{a^4-1}$ or $\dfrac{(a+1)(a^3+a^2+a+1)}{(a^2+1)(a+1)(a-1)}.$

§ 103.

3. $= \frac{-2}{b^2} = -2b^{-2}$.

5. $= -\frac{2a}{b} = -2ab^{-1}$.

6. $= 3a^2by$.

7. $= -\frac{2a^2}{b^2c} = -2a^2b^{-2}c^{-1}$.

8. $= \frac{4pqxy}{3bc} = 4.3^{-1}pqxyb^{-1}c^{-1}$.

9. $= \frac{3p^2x}{2} = 3.2^{-1}p^2x$.

10. $= \frac{4a^2(x-y)}{3} = 4.3^{-1}a^2(x-y)$

11. $= \frac{42b^2(x+y)^2}{(x-y)^2} \times \frac{(x-y)^2}{20(x+y)^2} = \frac{42b^2(x+y)}{20(x-y)} =$
$\frac{21b^2(x+y)}{10(x-y)} = 21.10^{-1}b^2(x+y)(x-y)^{-1}$.

12. $= \frac{22(a-b)(m-n)}{15(a+b)(m+n)} =$
$22.15^{-1}(a-b)(a+b)^{-1}(m-n)(m+n)^{-1}$.

13. $= \frac{25(a+b)(a-b)(m+n)(m-n)}{15(a-b)(m+n)} =$
$\frac{5(a+b)(m-n)}{3} = 5.3^{-1}(a+b)(m-n)$.

14. $= \frac{(x^2+1)(x^2-1)(a-2b)(a+2b)}{(x^2-1)(a+2b)} =$
$(x^2+1)(a-2b)$.

15. $= \frac{(x^2+1)(x^2-1)}{x^2+1} = x^2-1$.

16. $= \frac{xy^2}{a^2b} = xy^2a^{-2}b^{-1}$.

17. $= \frac{m^5n^2}{y^2z^5} = m^5n^2y^{-2}z^{-5}$.

18. $= \dfrac{(m+1)(m+2)(m+3)}{(m-1)(m-2)(m-3)} =$
$(m+1)(m+2)(m+3)(m-1)^{-1}(m-2)^{-1}(m-3)^{-1}$.

19. $= \dfrac{a^m}{a^n} = a^{m-n}$ or $a^m . a^{-n}$.

20. $= \dfrac{ab^m c^n}{qb^n c^m} = ab^{m-n}c^{n-m}q^{-1}$.

§ 104.

1. $\dfrac{abc}{abcd} = \dfrac{1}{d}$. $\dfrac{bcd}{abcd} = \dfrac{1}{a}$. $\dfrac{acd}{abcd} = \dfrac{1}{b}$. $\dfrac{abd}{abcd} = \dfrac{1}{c}$.

$\therefore \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} + \dfrac{1}{d}$. Ans.

2. $= -\dfrac{xyzu}{x^2y^2z^2u^2} = \dfrac{-1}{xyzu}$: $\dfrac{x^2yzu^2}{x^2y^2z^2u^2} = \dfrac{1}{yz}$:
$\dfrac{xy^2z^2u}{x^2y^2z^2u^2} = \dfrac{1}{xu}$; $-\dfrac{x^2y^2z^2u^2}{x^2y^2z^2u^2} = -1$.

$\therefore \dfrac{-1}{xyzu} + \dfrac{1}{yz} + \dfrac{1}{xu} - 1$. Ans.

3. $= \dfrac{a^2}{ab} = \dfrac{a}{b}$; $\dfrac{-b^2}{ab} = \dfrac{-b}{a}$. $\therefore \dfrac{a}{b} - \dfrac{b}{a}$. Ans.

4. $= \dfrac{a^2x}{ax} = a$; $\dfrac{-b^2y}{ax} = -\dfrac{b^2y}{ax}$. $\therefore a - \dfrac{b^2y}{ax}$. Ans.

5. $\dfrac{(m-n)(n+q)}{(m-n)(p-q)} = \dfrac{n+q}{p-q}$; $-\dfrac{(m+n)(p-q)}{(m-n)(p-q)} =$
$-\dfrac{m+n}{m-n}$. $\therefore \dfrac{n+q}{p-q} - \dfrac{m+n}{m-n}$. Ans.

6. $\dfrac{(x-a)(y-b)}{x^2-y^2} = \dfrac{(x-a)(y-b)}{x^2-y^2}$; $\dfrac{(x-y)(a-b)}{x^2-y^2} =$
$\dfrac{a-b}{x+y}$; $\dfrac{(x-b)(y-a)}{x^2-y^2} = \dfrac{(x-b)(y-a)}{x^2-y^2}$;
$\therefore \dfrac{(x-a)(y-b)}{x^2-y^2} + \dfrac{a-b}{x+y} + \dfrac{(x-b)(y-a)}{x^2-y^2}$. Ans.

7. $\dfrac{(a+b)(m-n)}{a^2-b^2}=\dfrac{m-n}{a-b}$; $-\dfrac{(a-b)(m+n)}{a^2-b^2}=$

$$-\frac{m+n}{a+b}.\quad \therefore \frac{m-n}{a-b}-\frac{m+n}{a+b}.\quad \textbf{Ans.}$$

§ 105.

1. $=\dfrac{a-ab+abc}{abc}=\dfrac{1-b+bc}{bc}$. **Ans.**

2. $=\dfrac{a-b}{(a-b)^2}=\dfrac{1}{a-b}$. **Ans.**

3. $=\dfrac{x-a+y-b+a+b+x-y}{a^2x}=\dfrac{2x}{a^2x}=\dfrac{2}{a^2}$. **Ans.**

4. $-\dfrac{d}{b-a}$ by changing signs of numerator and denominator becomes $\dfrac{d}{a-b}$, also $\dfrac{b}{b-a}=\dfrac{-b}{a-b}$; then

$$\frac{a}{a-b}+\frac{d}{a-b}=\frac{a+d}{a-b}.$$

$$\frac{-b}{a-b}-\frac{-c}{a-b}=\frac{-b-c}{a-b}.$$

$$\therefore \frac{a+d}{a-b}+\frac{-b-c}{a-b}=\frac{a-b-c+d}{a-b}.\quad \textbf{Ans.}$$

5. $-\dfrac{a-c}{m-n}$ by changing signs of numerator and the quotient $=+\dfrac{c-a}{m-n}$; then

$$\frac{a-b}{m-n}+\frac{c-a}{m-n}=\frac{c-b}{m-n}.$$

$-\dfrac{c-b}{n-m}$ changing signs of denominator and quotient $=\dfrac{c-b}{m-n}$; then

$$\frac{c-b}{m-n}+\frac{c-b}{m-n}=\frac{2c-2b}{m-n}.$$

Also changing signs of both numerator and denominator of $\dfrac{c+a}{n-m}$, we have $\dfrac{-c-a}{m-n}$.

$$\therefore \text{ then } \frac{2c-2b}{m-n}+\frac{-c-a}{m-n}=\frac{c-2b-a}{m-n}.\quad \textbf{Ans.}$$

§ 106.

1. L.C.D. $= x - 1$. $\qquad 1 = \dfrac{x-1}{x-1}$,

and adding $\dfrac{x-1}{x-1} + \dfrac{1}{x-1} = \dfrac{x}{x-1}$. Ans.

2. L.C.D. $\quad x + 1$. $\qquad 1 = \dfrac{x+1}{x+1}$,

and adding $\dfrac{x+1}{x+1} - \dfrac{1}{x+1} = \dfrac{x}{x+1}$. Ans.

3. L.C.D. $\quad (1-x)(1+x) = 1 - x^2$.

$\dfrac{1}{1-x} = \dfrac{1+x}{1-x^2}$; $\qquad -\dfrac{1}{1+x} = \dfrac{-(1-x)}{1-x^2} = \dfrac{x-1}{1-x^2}$.

$\therefore$ add. $\dfrac{1+x}{1-x^2} + \dfrac{x-1}{1-x^2} = \dfrac{2x}{1-x^2}$. Ans.

4. L.C.D. $\quad (1-x)(1+x) = 1 - x^2$.

$\dfrac{1}{1-x} = \dfrac{1+x}{1-x^2}$; $\qquad \dfrac{1}{1+x} = \dfrac{1-x}{1-x^2}$;

add. $\dfrac{1+x}{1-x^2} + \dfrac{1-x}{1-x^2} = \dfrac{2}{1-x^2}$. Ans.

5. L.C.D. $\quad a + x$. $\qquad x = \dfrac{ax + x^2}{a+x}$;

then add. $\dfrac{ax+x^2}{a+x} - \dfrac{ax}{a+x} - \dfrac{x^2}{a+x} = \dfrac{0}{a+x} = 0$. Ans.

6. L.C.D. $\quad (a-b)(a+b) = a^2 - b^2$.

$\dfrac{a}{a-b} = \dfrac{a^2+ab}{a^2-b^2}$; $-\dfrac{b}{a+b} = \dfrac{-(ab-b^2)}{a^2-b^2} = \dfrac{b^2-ab}{a^2-b^2}$;

add. $\dfrac{a^2+ab}{a^2-b^2} + \dfrac{b^2-ab}{a^2-b^2} = \dfrac{a^2+b^2}{a^2-b^2}$. Ans.

7. L.C.D. $\quad ax(a-x)$.

$\dfrac{a}{x(a-x)} = \dfrac{a^2}{ax(a-x)}$; $\dfrac{-x}{a(a-x)} = \dfrac{-x^2}{ax(a-x)}$;

add. $\dfrac{a^2}{ax(a-x)} - \dfrac{x^2}{ax(a-x)} = \dfrac{a^2-x^2}{ax(a-x)}$. Divide

both num. and den. by $a - x = \dfrac{a+x}{ax}$. Ans.

8. L.C.D. $x(4x^2-1)$.

$\frac{2x-5}{4x^2-1}=\frac{2x^2-5x}{x(4x^2-1)}$; $\frac{5}{2x-1}=\frac{10x^2+5x}{x(4x^2-1)}$; $\frac{-3}{x}=$

$\frac{-(12x^2-3)}{x(4x^2-1)}$; add. $\frac{2x^2-5x}{x(4x^2-1)}+\frac{10x^2+5x}{x(4x^2-1)}+$

$\frac{-12x^2+3}{x(4x^2-1)}=\frac{3}{x(4x^2-1)}$. Ans.

9. L.C.D. x^2-y^2.

$\frac{1}{x+y}=\frac{x-y}{x^2-y^2}$; $\frac{2y}{x^2-y^2}=\frac{2y}{x^2-y^2}$; $\frac{-1}{x-y}=$

$\frac{-x-y}{x^2-y^2}$; then add. $\frac{x-y}{x^2-y^2}+\frac{2y}{x^2-y^2}+\frac{-x-y}{x^2-y^2}=$

$\frac{0}{x^2-y^2}=0$. Ans.

10. L.C.D. $(a-b)(b-c)(c-a)=a^2c+ab^2-a^2b+bc^2-ac^2-b^2c$.

$\frac{1}{a-b}=\frac{bc-ab-c^2+ac}{a^2c+ab^2-a^2b+bc^2-ac^2-b^2c}$.

$\frac{1}{b-c}=\frac{ac-a^2-bc+ab}{a^2c+ab^2-a^2b+bc^2-ac^2-b^2c}$;

$\frac{1}{c-a}=\frac{ab-ac-b^2+bc}{a^2c+ab^2-a^2b+bc^2-ac^2-b^2c}$;

add. $\frac{bc-ab-c^2+ac+ac-a^2-bc+ab+ab-ac-b^2+bc}{\text{L.C.D.}}$.

$=\frac{bc+ab-c^2+ac-a^2-b^2}{a^2c+ab^2-a^2b+bc^2-ac^2-b^2c}=$

$\frac{bc+ab+ac-(a^2+b^2+c^2)}{(a-b)(b-c)(c-a)}$. Ans.

11. L.C.D. $(x+y)(x-y)=x^2-y^2$.

$\frac{a}{x+y}=\frac{ax-ay}{x^2-y^2}$; $\frac{a}{x-y}=\frac{ax+ay}{x^2-y^2}$;

add. $\frac{ax-ay}{x^2-y^2}+\frac{ax+ay}{x^2-y^2}=\frac{2ax}{x^2-y^2}$. Ans.

12. L.C.D. $(a-b)(a+b)=a^2-b^2$.

$\frac{a+b}{a-b}=\frac{a^2+2ab+b^2}{a^2-b^2}$; $\frac{-(a-b)}{a+b}=\frac{-(a^2-2ab+b^2)}{a^2-b^2}$

$=\frac{-a^2+2ab-b^2}{a^2-b^2}$;

add. $\frac{a^2+2ab+b^2}{a^2-b^2}+\frac{-a+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2}$. Ans.

13. L.C.D. $a^2 - b^2$. $\frac{a^2 + b^2}{a^2 - b^2} =$ same.

$\frac{-b}{a-b} = \frac{-ab - b^2}{a^2 - b^2}$; $\frac{a}{a+b} = \frac{a^2 - ab}{a^2 - b^2}$;

add. $\frac{a^2 + b^2}{a^2 - b^2} + \frac{-ab - b^2}{a^2 - b^2} + \frac{a^2 - ab}{a^2 - b^2} = \frac{2a^2 - 2ab}{a^2 - b^2} =$

$\frac{2a(a-b)}{a^2 - b^2}$, div. by $a - b = \frac{2a}{a+b}$. Ans.

14. L.C.D. $= 2x^2(x+1)(x-1) = 2x^2(x^2-1)$.

$\frac{1}{2(x-1)} = \frac{x^2(x+1)}{2x^2(x^2-1)} = \frac{x^3 + x^2}{2x^2(x^2-1)}$; $\frac{-1}{2(x+1)} =$

$\frac{-x^2(x-1)}{2x^2(x^2-1)} = \frac{-x^3 + x^2}{2x^2(x^2-1)}$; $-\frac{1}{x^2} = \frac{-2(x^2-1)}{2x^2(x^2-1)} =$

$\frac{-2x^2 + 2}{2x^2(x^2-1)}$;

add. $\frac{x^3 + x^2}{2x^2(x^2-1)} + \frac{-x^3 + x^2}{2x^2(x^2-1)} + \frac{-2x^2 + 2}{2x^2(x^2-1)} =$

$\frac{2}{2x^2(x^2-1)} = \frac{1}{x^2(x^2-1)}$. Ans.

15. $-\left(1 - \frac{b}{a-b}\right) = -1 + \frac{b}{a-b} = \frac{-a + b + b}{a-b}$;

add. $\frac{a}{a-b} + \frac{-a+b+b}{a-b} = \frac{2b}{a-b}$. Ans.

16. L.C.D. $(m-n)(x+y)$.

$\frac{m+n}{m-n} = \frac{(m+n)(x+y)}{(m-n)(x+y)} = \frac{mx + my + nx + ny}{(m-n)(x+y)}$

$\frac{-x+y}{x+y} = \frac{(-x+y)(m-n)}{(m-n)(x+y)} = \frac{-mx + my + nx - ny}{(m-n)(x+y)}$;

add. $\frac{mx + my + nx + ny}{(m-n)(x+y)} + \frac{-mx + my + nx - ny}{(m-n)(x+y)} =$

$\frac{2my + 2nx}{(m-n)(x+y)} = \frac{2(my + nx)}{(m-n)(x+y)}$. Ans.

17. L.C.D. $m^2(m-y)$.

$\frac{y}{m^2} = \frac{my - y^2}{m^2(m-y)}$; $\frac{-m-y}{m(m-y)} = \frac{-m^2 - my}{m^2(m-y)}$;

add. $\frac{my - y^2}{m^2(m-y)} + \frac{-m^2 - my}{m^2(m-y)} = \frac{-m^2 - y^2}{m^2(m-y)}$. Ans.

18. L.C.D. $a^2 - x^2$. $1 = \frac{a^2 - x^2}{a^2 - x^2}$.

$\frac{a}{a-x} = \frac{a^2 + ax}{a^2 - x^2}$; $\frac{x^2}{a^2 - x^2} =$ same;

add. $\frac{a^2 - x^2}{a^2 - x^2} - \frac{a^2 + ax}{a^2 - x^2} - \frac{x^2}{a^2 - x^2} = \frac{-2x^2 - ax}{a^2 - x^2} =$

$\frac{x(2x + a)}{x^2 - a^2}$. Ans.

19. L.C.D. $(a + b)(b + c)(c + a) =$
$b^2c + ac^2 + bc^2 + a^2b + ab^2 + a^2c + 2abc$.

$\frac{a-b}{a+b} = \frac{ac^2 + a^2b + a^2c - b^2c - bc^2 - ab^2}{\text{L.C.D.}}$

$\frac{b-c}{b+c} = \frac{b^2c + a^2b + ab^2 - ac^2 - bc^2 - a^2c + abc - abc}{\text{L.C.D.}}$;

$\frac{c-a}{c+a} = \frac{b^2c + ac^2 + bc^2 - a^2b - ab^2 - a^2c}{\text{L.C.D.}}$;

$\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)} = \frac{-b^2c - ac^2 + bc^2 - a^2b + ab^2 + a^2c}{\text{L.C.D.}}$;

add. $\frac{ac^2 + a^2b + a^2c - b^2c - bc^2 - ab^2}{\text{L.C.D.}} +$

$\frac{b^2c + a^2b + ab^2 - ac^2 - bc^2 - a^2c}{\text{L.C.D.}} +$

$\frac{b^2c + ac^2 + bc^2 - ab^2 - a^2b - a^2c}{\text{L.C.D.}} +$

$\frac{-b^2c - ac^2 + bc^2 - a^2b + ab^2 + a^2c}{\text{L.C.D.}} = \frac{0}{\text{L.C.D.}} = 0$. Ans.

20. L.C.D. $b(b - a)$. Since the $-$ sign changes the signs $-$, we can change it to have $(b - a)$.

$\frac{a}{b} = \frac{ab - a^2}{b(b-a)}$; $\frac{b}{b-a} = \frac{b^2}{b(b-a)}$; $\frac{a}{b-a} = \frac{ab}{b(b-a)}$;

add. $\frac{ab - a^2}{b(b-a)} + \frac{b^2}{b(b-a)} - \frac{ab}{b(b-a)} = \frac{b^2 - a^2}{b(b-a)}$.

Divide numerator and denominator by $b - a =$

$\frac{b+a}{b}$. Ans.

21. L.C.D. $x^2 - y^2$.

$\frac{m - x + a}{x + y} = \frac{(m - x + a)(x - y)}{x^2 - y^2} =$

$\frac{mx - x^2 + ax - my + xy - ay}{x^2 - y^2}$;

$$\frac{m-x-a}{x-y}=\frac{(m-x-a)(x+y)}{x^2-y^2}=$$
$$\frac{mx-x^2-ax+my-xy-ay}{x^2-y^2}.$$

Subtract
$$\frac{mx-x^2+ax-my+xy-ay}{x^2-y^2}-$$
$$\frac{mx-x^2-ax+my-xy-ay}{x^2-y^2}=$$
$$\frac{2ax-2my+2xy}{x^2-y^2}=\frac{2(ax-my+xy)}{x^2-y^2}.\quad \text{Ans.}$$

22. L.C.D. abc.
$$\frac{c}{ab}=\frac{c^2}{abc};\ \frac{a}{bc}=\frac{a^2}{abc};\ \frac{b}{ac}=\frac{b^2}{abc};$$
add. $\dfrac{c^2}{abc}+\dfrac{a^2}{abc}+\dfrac{b^2}{abc}=\dfrac{c^2+a^2+b^2}{abc}.$ Ans.

23.
$$\frac{b}{(b-a)(b-c)}=\frac{-b}{(a-b)(b-c)};$$
$$\frac{c}{(c-a)(c-b)}=\frac{-c}{(a-c)(c-b)}=\frac{c}{(a-c)(b-c)}.$$
L.C.D. $(a-b)(a-c)(b-c)$.
$$\frac{a}{(a-b)(a-c)}=\frac{ab-ac}{(a-b)(a-c)(b-c)};$$
$$\frac{-b}{(a-b)(b-c)}=\frac{-ab+bc}{(a-b)(a-c)(b-c)};$$
$$\frac{c}{(a-c)(b-c)}=\frac{ac-bc}{(a-b)(a-c)(b-c)};$$
add. $\dfrac{ab-ac}{\text{L.C.D.}}+\dfrac{-ab+bc}{\text{L.C.D.}}+\dfrac{ac-bc}{\text{L.C.D.}}=\dfrac{0}{\text{L.C.D.}}=0.$
Ans.

24. L.C.D. x^2-1. $\qquad \dfrac{x+1}{x-1}=\dfrac{x^2+2x+1}{x^2-1};$
$$\frac{-(x-1)}{x+1}=\frac{-(x^2-2x+1)}{x^2-1};\ 4x=\frac{4x^3-4x}{x^2-1};$$
add. $\dfrac{x^2+2x+1-x^2+2x-1+4x^3-4x}{x^2-1}=\dfrac{4x^3}{x^2-1}.$ Ans.

25. L.C.D. $a^2 - b^2$.

$$\frac{ab}{a+b} = \frac{a^2b - ab^2}{a^2 - b^2}; \quad \frac{-a^2}{a-b} = \frac{-a^3 - a^2b}{a^2 - b^2};$$

$$\frac{a(a^2+b^2)}{a^2-b^2} = \frac{a^3+ab^2}{a^2-b^2};$$

$$\text{add. } \frac{a^2b - ab^2 - a^3 - a^2b + a^3 + ab^2}{a^2 - b^2} = \frac{0}{a^2 - b^2} = 0. \quad \text{Ans.}$$

26. L.C.D. $x^2 - a^2$. $\quad 1 = \frac{x^2 - a^2}{x^2 - a^2};$

$$\frac{-a}{x+a} = \frac{-ax + a^2}{a^2 - x^2}; \quad \frac{-x}{x-a} = \frac{-x^2 - ax}{x^2 - a^2};$$

$$\text{add. } \frac{x^2 - a^2 + a^2 - ax - x^2 - ax}{x^2 - a^2} = \frac{-2ax}{x^2 - a^2}. \quad \text{Ans.}$$

27. L.C.D. $x^2 + y^2$. $\quad 1 = \frac{x^2 + y^2}{x^2 + y^2}; \quad \frac{-(x^2 - 2xy + y^2)}{x^2 + y^2} =$

$$\frac{-x^2 + 2xy - y^2}{x^2 + y^2};$$

$$\text{add. } \frac{x^2 + y^2 - x^2 + 2xy - y^2}{x^2 + y^2} = \frac{2xy}{x^2 + y^2}. \quad \text{Ans.}$$

28. L.C.D. $2ay$.

$$1 = \frac{2ay}{2ay}; \quad -\frac{a^2 + y^2 - x^2}{2ay} = \frac{-a^2 - y^2 + x^2}{2ay};$$

$$\text{add. } \frac{2ay - a^2 - y^2 + x^2}{2ay} \text{ or } \frac{-(a-y)^2 + x^2}{2ay}. \quad \text{Ans.}$$

29. L.C.D. $a^4 - 2a^2b^2 + b^4$. Since $(a+b)^2(a-b)^2 = a^4 - 2a^2b^2 + b^4$, and this is the square of $a^2 - b^2$,

$$\frac{1}{(a+b)^2} = \frac{a^2 - 2ab + b^2}{\text{L.C.D.}}; \quad \frac{1}{(a-b)^2} = \frac{a^2 + 2ab + b^2}{\text{L.C.D.}};$$

$$\frac{1}{a^2 - b^2} = \frac{a^2 - b^2}{\text{L.C.D.}};$$

$$\text{add. } \frac{a^2 - 2ab + b^2 + a^2 + 2ab + b^2 + a^2 - b^2}{a^4 - 2a^2b^2 + b^4} =$$

$$\frac{3a^2 + b^2}{a^4 - 2a^2b^2 + b^4}. \quad \text{Ans.}$$

30. L.C.D. $4ab$. $\quad 1 = \frac{4ab}{4ab};$

$$\text{add. } \frac{4ab + a^2 - 2ab + b^2}{4ab} = \frac{a^2 + 2ab + b^2}{4ab} =$$

$$\frac{(a+b)^2}{4ab}. \quad \text{Ans.}$$

§ 107.

1. $= \left(\frac{ab}{abc} - \frac{bc}{abc} - \frac{ac}{abc}\right) y = \left(\frac{1}{c} - \frac{1}{a} - \frac{1}{b}\right) y.$

2. $= u\left(\frac{mn}{mn} + \frac{mp}{mn} + \frac{pn}{mn}\right) = u\left(1 + \frac{p}{n} + \frac{p}{m}\right).$

3. $= \frac{ab(q+r) + bc(q+r)}{abc} = \frac{1}{c}\left(q + r\right) + \frac{1}{a}(q+r) =$
$\left(\frac{1}{c} + \frac{1}{a}\right)(q+r).$

4. $= \frac{a(x - 4y) - b(y + 3x)}{2am} = \frac{x - 4y}{2m} - \frac{b(y+3x)}{2am}.$

5. $= \frac{x(4m - 3a - 6c) + y(2 + a)}{xyz} = \frac{4m - 3a - 6c}{yz} +$
$\frac{2 + a}{xz}.$

6. $= \frac{a(a^2 + 2ab + b^2)}{xy} = \frac{a(a+b)^2}{xy}.$

7. $= \frac{a^2x - 4abc - 3ay + 4ac}{p+q} = \frac{a(ax - 4bc - 3y + 4c)}{p+q} =$
$\frac{a[ax + 4c(1 - b) - 3y]}{p+q}.$

8. $= \frac{x^2y - 4x - 2bx + 4cx - 3ax}{a+b} = \frac{x(xy - 4 - 2b + 4c - 3a)}{a+b}.$

9. $= \frac{ax^2 - 4cx - 3mx - 3am + 3mx + 3am}{2a - 3b} =$
$\frac{ax^2 - 4cx}{2a - 3b} = \frac{x(ax - 4c)}{2a - 3b}.$

10. $\frac{4a\sqrt{x} - 2c\sqrt{x} + 2b\sqrt{x} - 2mn\sqrt{x} + 8\sqrt{x}}{3a - 4b} =$
$\frac{2\sqrt{x}(2a - c + b - mn + 4)}{3a - 4b}.$

§ 108.

1. Cancel $x - a = ab + y$. Ans.

Cancel x and $a = b$. Ans.

3. Cancel $x = -aby$. Ans.

4. Cancel $x - a = ac(x + a) = acx + a^2c$. Ans.

5. Cancel $xy = \dfrac{abmy}{x}$. Ans.

6. $$ax^2 + \frac{m-a}{x-m} = \frac{ax^4 - ax^3m + m - a}{x-m} \times \frac{m}{x^2} = \frac{m(ax^4 - ax^3m + m - a)}{x^2(x-m)}. \text{ Ans.}$$

7. $$= \frac{(a-b)(a+b)}{m \times m} = \frac{a^2 - b^2}{m^2}. \text{ Ans.}$$

8. $$a + \frac{m}{n} = \frac{an+m}{n}. \quad n + \frac{n}{m} = \frac{nm+n}{m}.$$
$$\frac{an+m}{n} \times \frac{nm+n}{m} \text{ cancel } n = \frac{(an+m)(m+1)}{m}. \text{ Ans.}$$

9. $$ab - \frac{x}{y} = \frac{aby-x}{y}. \quad ay + \frac{y-ab}{x} = \frac{axy+y-ab}{x}.$$
$$\frac{aby-x}{y} \times \frac{axy+y-ab}{x} = \frac{(aby-x)(axy+y-ab)}{xy}. \text{ Ans.}$$

10. Cancel $(m+n) = \dfrac{n-m}{m-n}$. Ans.

11. $$a + \frac{bx}{m} = \frac{am+bx}{m}. \quad \frac{a}{b} + \frac{b}{x} + \frac{x}{a} = \frac{a^2x + b^2a + bx^2}{abx}.$$
$$\frac{am+bx}{m} \times \frac{a^2x + ab^2 + bx^2}{abx} = \frac{(am+bx)(a^2x+ab^2+bx^2)}{abmx}. \text{ Ans.}$$

12. $$m + \frac{mn}{m-n} = \frac{m^2 - mn + mn}{m-n} = \frac{m^2}{m-n}.$$
$$m - \frac{mn}{m+n} = \frac{m^2 + mn - mn}{m+n} = \frac{m^2}{m+n}.$$
$$\frac{m^2}{m-n} \times \frac{m^2}{m+n} = \frac{m^4}{m^2 - n^2}. \text{ Ans.}$$

13. $a - \frac{bx}{a} = \frac{a^2 - bx}{a}$. $b - \frac{ax}{b} = \frac{b^2 - ax}{b}$.

$\frac{a^2 - bx}{a} \times \frac{b^2 - ax}{b} = \frac{(a^2 - bx)(b^2 - ax)}{ab} =$

$\frac{a^2b^2 - a^3x - b^3x + abx^2}{ab} = ab + x^2 - x\frac{a^3 + b^3}{ab}$. Ans.

14. $b - \frac{bx}{a} = \frac{ab - bx}{a}$.

$\frac{ab - bx}{a} \times \frac{a}{x}$ cancel $a = \frac{ab - bx}{x} = b\left(\frac{a}{x} - 1\right)$. Ans.

15. Multiply denominator by $p = \frac{m}{np}$. Ans.

16. Multiply denominator by $a + b = \frac{a}{a^2 - b^2}$. Ans.

17. Multiply denominator by $x - 1 = \frac{x - a}{x^2 - 1}$. Ans.

18. Multiply denominator by $1 + x^2 = \frac{a + b}{x^4 - 1}$. Ans.

19. $\frac{-2a - 3m}{a^n + b^n} = -\frac{2a + 3m}{a^n + b^n}$.

Multiply denominator by $b^n - a^n = -\frac{2a + 3m}{b^{2n} - a^{2n}} =$

$\frac{2a + 3m}{a^{2n} - b^{2n}}$. Ans.

§ 110.

1. Multiply both terms by y gives $\frac{y + x}{y - x}$. Ans.

Suppose we multiply the numerators by y,

we get $\frac{y + \frac{xy}{y}}{y - \frac{xy}{y}} = \frac{\frac{y^2 + xy}{y}}{\frac{y^2 - xy}{y}}$;

taking y from both denominators gives $\frac{y^2 + xy}{y^2 - xy}$;

dividing by y gives $\frac{y + x}{y - x}$, the answer as above.

2. Multiply both terms by $x = \dfrac{ax+b}{ax-b}$. Ans.

3. Multiply both terms by $a^2 - x^2 = \dfrac{a^2 - 2ax + x^2}{a^2 + 2ax + x^2}$. Ans.

4. Multiply both terms by $kmn = \dfrac{abk}{bdn} = \dfrac{ak}{dn}$. Ans.

5. Multiply both terms by $n+1 = \dfrac{n+1+n-1}{n+1-(n-1)} =$
$\dfrac{n+1+n-1}{n+1-n+1} = \dfrac{2n}{2} = n$. Ans.

6. Multiply both terms by $1 - x^2 =$
$\dfrac{1+2x+x^2+1-2x+x^2}{1+2x+x^2-(1-2x+x^2)} = \dfrac{1+2x+x^2+1-2x+x^2}{1+2x+x^2-1+2x-x^2} =$
$\dfrac{2+2x^2}{4x} = \dfrac{2(1+x^2)}{4x} = \dfrac{1+x^2}{2x}$. Ans.

7. Multiply both terms by $mn =$
$\dfrac{am^2n+bn}{amn^2-bm} = \dfrac{n(am^2+b)}{m(an^2-b)}$. Ans.

8. Multiply both terms by $y = \dfrac{2xy-3}{y(a+b-x)}$. Ans.

9. Multiply both terms by $4ab = \dfrac{4ab+(a-b)^2}{4ab-2(b^2-a^2)} =$
$\dfrac{4ab+a^2-2ab+b^2}{4ab-2b^2+2a^2} = \dfrac{a^2+2ab+b^2}{2a^2+4ab-2b^2} =$
$\dfrac{(a+b)^2}{2(a^2+2ab-b^2)}$. Ans.

10. Multiply both terms by $1 - a^2 = \dfrac{1-a+a+a^2}{1+a-a+a^2} =$
$\dfrac{1+a^2}{1+a^2} = 1$. Ans.

11. Multiply both terms by $a^2 = \dfrac{a^4+1+2a^2}{a+a^3} =$
$\dfrac{(a^2+1)^2}{a(1+a^2)} = \dfrac{a^2+1}{a} = a + \dfrac{1}{a}$. Ans.

12. Multiply both terms by $ab^3 = \dfrac{a^3+b^3}{b^3(a^2-a+b)}$. Ans.

13. Multiply both terms by $b(a+b) = \dfrac{ab+a^2+2b^2+ab}{a^2+3ab+2b^2-ab} = \dfrac{2ab+2b^2+a^2}{2ab+2b^2+a^2} = 1$. Ans.

14. Numerator $= \dfrac{x-y}{x+y} - \dfrac{x+y}{x^2-y^2}$. Multiply by x^4-y^4 and we get $\dfrac{(x-y)^2(x^2+y^2)-(x+y)(x^2+y^2)}{(x+y)^2(x^2+y^2)-(x^2-y^2)}$. Ans.

§ 111.

1. $\dfrac{ab}{a-b} \times \dfrac{b}{a}$ cancel $a = \dfrac{b^2}{a-b}$. Ans.

2. $\dfrac{x+1}{8} \times \dfrac{9}{2x} = \dfrac{9x+9}{16x}$. Ans.

3. $\dfrac{x}{x-1} \times \dfrac{2}{x}$ cancel $x = \dfrac{2}{x-1}$. Ans.

4. $\dfrac{a^4-b^4}{a^2-2ab+b^2} \times \dfrac{a-b}{a(a+b)}$ cancel $(a-b) = \dfrac{a^4-b^4}{a-b} \times \dfrac{1}{a(a+b)}$. Since $\dfrac{a^4-b^4}{a-b} = \dfrac{(a^2+b^2)(a+b)(a-b)}{a-b} \times \dfrac{1}{a(a+b)}$ cancel $(a-b)$ and $(a+b) = \dfrac{a^2+b^2}{a}$. Ans.

5. $\dfrac{x+1}{x-1} \times \dfrac{x^2-1}{x+1}$ cancel $x+1 = \dfrac{x^2-1}{x-1}$; divide numerator and denominator by $x-1 = x+1$. Ans.

6. $\dfrac{a}{b} + \dfrac{m}{n} = \dfrac{an+bm}{bn}$; $\dfrac{b}{a} - \dfrac{n}{m} = \dfrac{bm-an}{am}$;

$\dfrac{an+bm}{bn} \times \dfrac{am}{bm-an} = \dfrac{a^2mn+abm^2}{b^2mn-abn^2} =$

$\dfrac{am(an+bm)}{bn(bm-an)}$. Ans.

7. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{ayz+bxz+cxy}{xyz}$;

$\frac{m}{x}+\frac{n}{y}+\frac{p}{z}=\frac{myz+nxz+pxy}{xyz}$;

$\frac{ayz+bxz+cxy}{xyz}\times\frac{xyz}{myz+nxz+pxy}$ cancel xyz $=$

$\frac{ayz+bxz+cxy}{myz+nxz+pxy}$. Ans.

8. $\frac{a}{a-b}-\frac{b}{a+b}=\frac{a^2+ab-ab+b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}$;

$\frac{b}{a-b}+\frac{a}{a+b}=\frac{ab+b^2+a^2-ab}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}$;

$\frac{a^2+b^2}{a^2-b^2}\times\frac{a^2-b^2}{a^2+b^2}$. Cancel a^2+b^2 and $a^2-b^2=1$. Ans.

§ 112.

1. The denominator must be less than .002.
2. The denominator must be less than .000 002.
3. The denominator must be less than .000 000 000 2.

§ 123.

1. L.C.D. $=$ 9 gives $2x-54=0$.
2. L.C.D. $=35$ gives $7x-5x=2450$.
3. L.C.D. $=12$ gives $6x+4x-3x=60$.
4. L.C.D. $=a^2$ gives $ax+x^2=ab$.
5. L.C.D. $=a^2b^2$ gives $abx+ab^2y+7a^2b=c$.
6. L.C.D. $=60$ gives $5(4a+3b)=12x$.
7. L.C.D. $=x^2-a^2$ gives $x^2+ax-x^2+ax=x^2-a^2$ or $2ax=x^2-a^2$.
8. L.C.D. $=(x-a)(x+b)$ gives $x^2+bx=2x^2-2ax$.
9. L.C.D. $=x-a$ gives $x+a=x^2+2ax$.
10. L.C.D. $=x^2-25$ gives $x^2+3x-10=x^2-3x-10$.

11. L.C.D. $= bxy$ gives $bx^2 - by^2 = axy$.

12. L.C.D. $= a(x^2 - a^2)$ gives $ax^2 - 2a^2x + a^3 - ax^2 - 2a^2x - a^3 + x^3 - a^2x = 0$ or $x^3 - 5a^2x = 0$.

13. L.C.D. $= a - b$ gives $x - y = az - bz$.

14. L.C.D. $= x - a$ gives $x + a = x - b$.

§ 124.

1. Clear of fractions $= 6y^2 + 4y = 7y - 49$,
transpose $= 6y^2 - 3y + 49 = 0$. Ans.

2. Clear of fractions $= x^2 - ax = x^2 + 2ax + a^2$,
transpose $= 3ax + a^2 = 0$. Ans.

3. Clearing of fractions $= 4x^2 - 30x + 14 = 4x^2 + 32x + 60$,
transpose $= -62x - 46 = 0$. $\div -2$ gives $31x + 23 = 0$.

4. C. of F. $= 2x^4 - 6a^2x^2 + 4a^3x - ax^3 + 3a^3x - 2a^4 - 4x^4 - 20ax^3 + a^2x^2 + 5a^3x = 14x^4 - 10ax^3 + 7ax^3 - 5a^2x^2$,
trans. $= -16x^4 - 18ax^3 + 12a^3x - 2a^4 = 0$ or $\div$ by -2,
we get $8x^4 + 9ax^3 - 6a^3x + a^4 = 0$. Ans.

5. C. of F. $= a^4y^3 + a^2y^4 + 2a^4y^3 - 2a^2y^4 + 3a^3y - 3y^3 = 7a^4y^2 - 7a^2y^4$,
trans. $= 6a^2y^4 + (3a^4 - 3)y^3 - 7a^4y^2 + 3a^3y = 0$,
or div. by $y = 6ay^3 + 3(a^4 - 1)y^2 - 7a^4y + 3a^3 = 0$.

6. C. of F. $= abz + az^2 + bz^2 + z^3 + a^3 + a^2z + a^2b + abz + ab^2 + abz + b^3 + b^2z = 0$.
Collect. $= z^3 + (a + b)z^2 + (a^2 + 3ab + b^2)z + a^3 + b^3 + a^2b + ab^2 = 0$. Ans.

7. By transposing second member and changing the sign of the fraction and the sign of the denominator, we get
$$\frac{z^2}{a - z} + \frac{z^3}{a^2 - z^2} + \frac{a^2z}{a^2 - z^2} = 0.$$
C. of F. $az^2 + z^3 + z^3 + a^2z = 0$.
Collect. $2z^3 + az^2 + a^2z = 0$. Ans.

8. C. of fractions $= 7y^3 + 6y^2 + 5y + 4 = 0$. Ans.

9. C. of F. L.C.D. $x^4 - a^4 = ax^3 + a^3x + a^2x^2 + a^4 + a^2x^2$
$+ a^4 + a^4 = x^4 - a^4$,
transpose $= x^4 - ax^3 - 2a^2x^2 - a^3x - 4a^4 = 0$. **Ans.**

10. $\div$ by $b = \dfrac{1}{c-z} + \dfrac{b}{c^2-z^2} + \dfrac{b^2}{c^4-z^4} = \dfrac{b^5}{c^6-z^6}$;
$\times$ by $c^4 - z^4$;
$= c^3 + c^2z + cz^2 + z^3 + bc^2 + bz^2 + b^2 = \dfrac{b^5(c^4-z^4)}{c^6-z^6}$
$= z^3 + bz^2 + cz^2 + c^2z + b^2 + bc^2 + c^3 = \dfrac{b^5(c^2+z^2)(c^2-z^2)}{c^6-z^6}$
$= \dfrac{b^5(c^2+z^2)}{c^4+c^2z^2+z^4}$
$= c^4z^3 + c^2z^5 + z^7 + bc^4z^2 + bc^2z^4 + bz^6 + c^5z^2 + c^3z^4 + cz^6 + c^6z$
$+ c^4z^3 + c^2z^5 + b^2c^4 + b^2c^2z^2 + b^2z^4 + bc^6 + bc^4z^2 + bc^2z^4 + c^7$
$+ c^5z^2 + c^3z^4 = b^5c^2 + b^5z^2$,
trans. $z^7 + (b+c)z^6 + 2c^2z^5 + (b^2 + 2bc^2 + 2c^3)z^4 + 2c^4z^3$
$+ (2c^5 + 2bc^4 + b^2c^2 - b^5)z^2 + c^6z + c^7 + bc^6 + b^2c^4 - b^5c^2 = 0$.

11. $= \dfrac{ax}{bx-1} = \dfrac{b}{x-a}$.
Clear of fractions $= ax^2 - a^2x = b^2x - b$;
transpose $= ax^2 - (a^2 + b^2)x - b = 0$. **Ans.**

12. $\dfrac{m}{nx - \dfrac{n}{x}} = \dfrac{mx}{nx^2 - n}$. $\qquad \dfrac{m}{x + \dfrac{1}{x}} = \dfrac{mx}{x^2+1}$;

then $\dfrac{mx}{nx^2 - n} = \dfrac{mx}{x^2+1}$ divide numerator by $mx =$

$\dfrac{1}{mx^2 - n} = \dfrac{1}{x^2+1}$.
Clear of fractions $x^2 + 1 = nx^2 - n$,
transpose $= x^2(1-n) + n + 1 = 0$. **Ans.**

13. $\dfrac{a}{a - \dfrac{1}{x}} = \dfrac{ax}{ax-1}$. $\qquad \dfrac{a^2}{a^2 - \dfrac{1}{x^2}} = \dfrac{a^2x^2}{a^2x^2 - 1}$;

then $\dfrac{ax}{ax-1} + \dfrac{a^2x^2}{a^2x^2-1} = \dfrac{a^3}{x^3}$.
Cl. of fr. L.C.D. $= (a^2x^2 - 1)x^3 = a^2x^5 + ax^4 + a^2x^5 =$
$a^5x^2 - a^3$,
transpose $= 2a^2x^5 + ax^4 - a^5x^2 + a^3 = 0$. **Ans.**

14. $\frac{3z}{z+\frac{1}{2}} = \frac{6z}{2z+1}$. $\frac{5z^2}{3z - \frac{3}{z}} = \frac{5z^3}{3z^2-3}$;

then $\frac{6z}{2z+1} - \frac{5z^3}{3z^2-3} = \frac{1}{z}$.

Cl. of fr. $18z^4 - 18z^2 - 10z^5 - 5z^4 = 6z^3 - 6z + 3z^2 - 3$,

trans. $-10z^5 + 13x^4 - 21z^2 - 6z^3 + 6z + 3 = 0$. Ans.

15. $\frac{ax}{1 - \frac{1}{x+a}} = \frac{ax^2 + a^2x}{x+a-1}$. $\frac{bx}{1 + \frac{1}{x-a}} = \frac{bx^2 - abx}{x-a+1}$;

then $\frac{ax^2 + a^2x}{x+a-1} = \frac{bx^2 - abx}{x-a+1}$.

Cl. of fr. $ax^3 + a^2x^2 - a^2x^2 - a^2x + ax^2 + a^2x = bx^3 - abx^2$
$+ abx^2 - a^2bx - bx^2 + abx$;

$ax^3 - a^2x + ax^2 + a^2x = bx^3 - a^2bx - bx^2 + abx$.

Transpose $= (a-b)x^3 + (a+b)x^2 + (a^2 + a^2b - a^2 - ab)x$
$= 0$. Ans.

16. $\frac{\frac{a}{x} - \frac{b}{a-x}}{\frac{b}{x}} = \frac{a^2 - ax - bx}{ab - bx}$; $\frac{a}{a - \frac{b}{x}} = \frac{ax}{ax-b}$.

$\frac{a^2 - ax - bx}{ab - bx} = \frac{ax}{ax-b}$. Cl. of fr.

$a^2x - a^2x^2 - abx^2 - a^2b + abx + b^2x = a^2bx - abx^2$,

trans. $= a^2x^2 + (-a^2 - ab - b^2 + a^2b)x + a^2b = 0$. Ans.

§ 129.

1. Clear of fractions $= 15 - 9x = 16x - 18$,

transpose $= -25x = -33$;

$\therefore x = \frac{-33}{-25} = \frac{33}{25}$. Ans.

2. Dividing by -1, the coefficient of x, we get

$x = \frac{a}{-1} = -a$. Ans.

3. Clear of fractions $= 6x + 3x + 2x = 132$.

Collect. $= 11x = 132$;

$\therefore x = \frac{132}{11} = 12$. Ans.

4. Clear of fractions $= x + 23 = 9x - 9$,
transpose $= -8x = -32$.
$$\therefore x = \frac{-32}{-8} = 4. \quad \text{Ans.}$$

5. Clear of fractions $= bcy + acy - aby = abc$.
Collect. $= (bc + ac - ab)\,y = abc$.
$$\therefore y = \frac{abc}{bc + ac - ab}. \quad \text{Ans.}$$

6. Clear of fractions $= 36u = 45u - 225$,
transpose $= -9u = -225$.
$$\therefore u = \frac{-225}{-9} = 25. \quad \text{Ans.}$$

7. Clear of fractions $20u - 15u + 12u = 60u - 1560$,
transpose $= -43u = -1560$.
$$\therefore u = \frac{-1560}{-43} = \frac{1560}{43}. \quad \text{Ans.}$$

8. Transpose $= -bx - ax = b - a$, div. by -1, we get
$bx + ax = a - b$,
or $(a + b)\,x = a - b$.
$$\therefore x = \frac{a - b}{a + b}. \quad \text{Ans.}$$

9. Clear of fractions $= bu + au = b + a$,
or $(b + a)\,u = b + a$.
$$\therefore u = \frac{b + a}{b + a} = 1. \quad \text{Ans.}$$

10. Clear of fractions $= 9x + 3 - x = 3x$,
transpose $5x = -3$.
$$\therefore x = -\frac{3}{5}. \quad \text{Ans.}$$

11. Clear of fractions $a^2 - ax = c^2 - cx$,
transpose $= cx - ax = c^2 - a^2$ or $(c - a)\,x = c^2 - a^2$.
$$\therefore x = \frac{c^2 - a^2}{c - a} = c + a. \quad \text{Ans.}$$

12. Clear of fractions, each member separately.

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x^2-4x+3-(x^2-4x+4)}{x^2-5x+6}=$$

$$\frac{x^2-4x+3-x^2+4x-4}{x^2-5x+6}=\frac{-1}{x^2-5x+6}; \text{ again,}$$

$$\frac{x-5}{x-6}-\frac{x-6}{x-7}=\frac{x^2-12x+35-(x^2-12x+36)}{x^2-13x+42}=$$

$$\frac{x^2-12x+35-x^2+12x-36}{x^2-13x+42}=\frac{-1}{x^2-13x+42};$$

then $\frac{-1}{x^2-5x+6}=\frac{-1}{x^2-13x+42}$.

Clear of fractions $-x^2+13x-42=-x^2+5x-6$,

transpose $=8x=36$.

$$\therefore x=\frac{36}{8}=\frac{9}{2}. \quad \text{Ans.}$$

13. Divide by $-1=y=\frac{a-b}{-1}=b-a$. Ans.

14. Clear of fractions, each side separately.

$$\frac{1}{x-2}-\frac{1}{x-4}=\frac{x-4-(x-2)}{x^2-6x+8}=\frac{x-4-x+2}{x^2-6x+8}=\frac{-2}{x^2-6x+8},$$

also $\frac{1}{x-6}-\frac{1}{x-8}=\frac{x-8-(x-6)}{x^2-14x+48}=\frac{x-8-x+6}{x^2-14x+48}=$

$$\frac{-2}{x^2-14x+48};$$

then $\frac{-2}{x^2-6x+8}=\frac{-2}{x^2-14x+48}$.

Cl. of fr. $-2x^2+28x-96=-2x^2+12x-16$,

trans. $=-2x^2+28x+2x^2-12x=96-16$,

or

$$16x=80.$$

$$\therefore x=\frac{80}{16}=5. \quad \text{Ans.}$$

15. Remove parentheses.

$$\frac{x}{2}-\frac{a}{6}-\frac{x}{3}+\frac{a}{12}+\frac{x}{4}-\frac{a}{20}=0.$$

Cl. of fr. $30x-10a-20x+5a+15x-3a=0$,

trans. $=30x-20x+15x=10a-5a+3a$.

$$25x=8a.$$

$$\therefore x=\frac{8a}{25}. \quad \text{Ans.}$$

16. Cl. of fr. $= b^2u - a^2u + abu + a^2u = a^2b - a^3$,

$$(b^2 + ab)\,u = a^2b - a^3 = a^2(b - a).$$

$$\therefore u = \frac{a^2(b-a)}{b^2+ab} = \frac{a^2(b-a)}{b(b+a)}. \quad \text{Ans.}$$

17. Clear of fractions. $a^2bx + ab^2 = bx + a$,

transpose $= a^2bx - bx = a - ab^2$,

$$(a^2b - b)\,x = a - ab^2 = a(1 - b^2).$$

$$\therefore x = \frac{a(1-b^2)}{a^2b-b} = \frac{a(1-b^2)}{b(a^2-1)}. \quad \text{Ans.}$$

18. Cl. of fr. $= acu - a^2c + abu - ab^2 + bcu - bc^2 = u - a - b - c$,

transpose $= acu + abu + bcu - u = a^2c + ab^2 + bc^2 - a - b - c$.

$$(ac + ab + bc - 1)\,u = a^2c + ab^2 + bc^2 - a - b - c.$$

$$\therefore u = \frac{a^2c + ab^2 + bc^2 - a - b - c}{ac + ab + bc - 1} =$$

$$\frac{a(ac + b^2 - 1) + b(c^2 - 1) - c}{a(c + b) + bc - 1}. \quad \text{Ans.}$$

19. Cl. of fr. $= m(x^2 + 2ax + a^2) + n(x^2 + 2bx + b^2) = mx^2$

$+ max + mbx + mab + nx^2 + nax + nbx + nab$,

or $mx^2 + 2amx + a^2m + nx^2 + 2bnx + b^2n = mx^2$

$+ max + mbx + mab + nx^2 + nax + nbx + nab$,

trans. $= mx^2 - mx^2 + 2amx - amx + nx^2 - nx^2 + 2bnx - bnx$

$- mbx - nax = mab + nab - a^2m - b^2n$,

$$(am + bn - mb - an)\,x = mab + nab - a^2m - b^2n.$$

$$\therefore x = \frac{mab + nab - a^2m - b^2n}{am + bn - mb - an} = \frac{bn(a-b) - am(a-b)}{m(a-b) - n(a-b)} =$$

$$\frac{bn - am}{m - n}. \quad \text{Ans.}$$

20. $(x - a)^3 = x^3 - 3x^2a + 3xa^2 - a^3$.

$(x - b)^3 = x^3 - 3x^2b + 3xb^2 - b^3$.

$(x - c)^3 = x^3 - 3x^2c + 3xc^2 - c^3$.

$3(x - a)(x - b)(x - c) = 3x^3 - 3bx^2 - 3ax^2 + 3abx - 3cx^2$

$+ 3bcx + 3acx - 3abc$;

then $3x^3 - 3x^2a - 3x^2b - 3x^2c + 3xa^2 + 3xb^2 + 3xc^2 - a^3 - b^3$

$- c^3 = 3x^3 - 3bx^2 - 3ax^2 + 3abx - 3cx^2 + 3bcx + 3acx - 3abc$,

trans. $3x^3 - 3x^3 - 3x^2a + 3x^2a - 3x^2b + 3bx^2 - 3x^2c + 3x^2c$

$+ 3xb^2 + 3xa^2 + 3xc^2 - 3abx - 3bcx - 3acx = a^3 + b^3 + c^3 - 3abc$,

or $(3a^2 + 3b^2 + 3c^2 - 3ab - 3bc - 3ac)x = a^3 + b^3 + c^3 - 3abc$.

$$\therefore x = \frac{a^3 + b^3 + c^3 - 3abc}{3(a^2 + b^2 + c^2 - ab - bc - ac)}. \quad \text{Ans.}$$

21. Clear of fractions, $= ab - ad + bd - cd = 0$.

1st, to find a, trans. $ab - ad = cd - bd$.

$$a(b-d) = d(c-b).$$

$$\therefore a = \frac{d(c-b)}{b-d}. \quad \text{Ans.}$$

2d, to find b, trans. $ab + bd = cd + ad$.

$$b(a+d) = d(c+a).$$

$$\therefore b = \frac{d(c+a)}{a+d}. \quad \text{Ans.}$$

3d, to find c,

trans. $= -cd = ad - ab - bd$.

$$\therefore c = \frac{ad - ab - bd}{-d} = \frac{ab + bd - ad}{d}. \quad \text{Ans.}$$

4th, to find d,

trans. $bd - ad - cd = -ab$.

$$d(b-a-c) = -ab.$$

$$\therefore d = \frac{-ab}{b-a-c} = \frac{ab}{a+c-b}. \quad \text{Ans.}$$

22. Clear of fractions, $ab + cd = 0$.

To find a, trans. $ab = -cd$.

$$a = \frac{-cd}{b}. \quad \text{Ans.}$$

Find b, $ab = -cd$.

$$b = \frac{-cd}{a}. \quad \text{Ans.}$$

Find c, transpose $cd = -ab$.

$$c = \frac{-ab}{d}. \quad \text{Ans.}$$

Find d, $cd = -ab$.

$$d = \frac{-ab}{c}. \quad \text{Ans.}$$

§ 130.

1. Let $x =$ the number.

Multiply by $4 = 4x$. Subtract from $100 = 100 - x$.

$$\therefore 4x = 100 - x.$$

$$5x = 100.$$

$$x = 20. \quad \text{Ans.}$$

2. Let $x =$ the number.

Divide by $8 = \frac{x}{8}$. Subtract from $81 = 81 - x$.

$$\therefore \frac{x}{8} = 81 - x.$$

$$\begin{aligned} \text{Clear of fractions, } x &= 648 - 8x, \\ \text{transpose } 9x &= 648. \\ x &= 72. \quad \text{Ans.} \end{aligned}$$

3. Let

$$\begin{aligned} x &= \text{share of the 1st;} \\ \text{then } 3x + 16 &= \text{share of the 2d;} \\ \text{then } x + 3x + 16 &= 284, \\ \text{transpose} = 4x &= 268. \\ x &= \$67, \text{ 1st,} \\ 3x + 16 &= 217, \text{ 2d.} \end{aligned} \Bigg\} \text{ Ans.}$$

4. Let $x =$ number;

then $\frac{x}{5} = \frac{1}{5}$;

$\frac{x}{7} = \frac{1}{7}$; then $\frac{x}{5} - \frac{x}{7} = 12$.

$$\begin{aligned} \text{Clear of fractions, } 7x - 5x &= 420. \\ 2x &= 420. \\ x &= 210. \quad \text{Ans.} \end{aligned}$$

5. Let $x =$ the number of sheep;

$$\begin{aligned} 10 \text{ more} &= 10 + x; \\ \text{then } (x + 10)\,5 &= 6x. \\ 5x + 50 &= 6x. \\ -x &= -50 \div \text{by} -1. \\ x &= 50. \quad \text{Ans.} \end{aligned}$$

6. Let

$$\begin{aligned} x &= \text{number of apples;} \\ (x - 60) &= \text{number of good apples;} \\ 2 \text{ for } 3 \text{ cents} &= \tfrac{3}{2} \text{ cents each;} \\ (x - 60) \text{ at } \tfrac{3}{2} \text{ cents each} &= \tfrac{3}{2}(x - 60). \\ x \text{ apples at } 1 \text{ cent each} &= x \text{ cents.} \end{aligned}$$

$\therefore \frac{3}{2}(x - 60) = x.$ Cl. of fr. $3(x - 60) = 2x,$

$$\begin{aligned} 3x - 180 &= 2x, \\ \text{transpose } x &= 180. \quad \text{Ans.} \end{aligned}$$

7. Let $x =$ present age;
$x + 10 =$ age 10 years hence;
$x - 20 =$ age 20 years ago;
$x - 26 =$ age 26 years ago.

$\frac{x+10}{x-20} = \frac{x}{x-26}$. Cl. of fr. $x^2 - 16x - 260 = x^2 - 20x$,

transpose $x^2 - x^2 - 16x + 20x = 260$.

$4x = 260$.

$x = 65$. Ans.

8. Let $x =$ A's share;
$x - 20 =$ B's share.
$x + x + 20 - 20$ or $2x =$ C's share;
then $x + x - 20 + 2x = 500$,
transpose $= 4x = 500 + 20 = 520$.

$$\left.\begin{aligned} x &= \frac{520}{4} = \$130, \text{ A},\\ x - 20 &= \quad 110, \text{ B},\\ 2x &= \quad 260, \text{ C}.\end{aligned}\right\} \text{Ans.}$$

9. Let $x =$ share of the youngest;
$x + \ \ 500 =$ " " next younger;
$x + 1000 =$ " " " "
$x + 1500 =$ " " " "
$x + 2000 =$ " " " "
then $x + x + 500 + x + 1000 + x + 1500 + x + 2000 = 10000$,
or $5x + 5000 = 10000$,
trans. $5x = 10000 - 5000 = 5000$.

$$\left.\begin{aligned} x &= 1000, \text{ 1st},\\ x + 500 &= 1500, \text{ 2d},\\ x + 1000 &= 2000, \text{ 3d},\\ x + 1500 &= 2500, \text{ 4th},\\ x + 2000 &= 3000, \text{ 5th}.\end{aligned}\right\} \text{Ans.}$$

10. Let $x =$ wife's age at time of marriage;
$x + \ \ 6 =$ husband's age at time of marriage;
$x + 12 =$ wife's age 12 years after marriage;
$x + 18 =$ husband's age 12 years after marriage;
then $8(x + 12) = 7(x + 18)$,
or $8x + 96 = 7x + 126$;

$$\left.\begin{aligned} \text{trans. } x &= 126 - 96 = 30, \text{ wife},\\ x + 6 &= 36, \text{ husband}.\end{aligned}\right\} \text{Ans.}$$

11. Let $x =$ age of the youngest;
$x+8 =$ " " " next younger;
$x+x+8 =$ " " " oldest;
$x+10 =$ " " " 1st 10 years hence;
$x+18 =$ " " " 2d 10 " "
$2x+18 =$ " " " 3d 10 " "
then $x+10+x+18+2x+18=120$,
or $4x+46=120$;
transpose $4x = 120-46 = 74$.

$$\left.\begin{array}{l} x = \frac{74}{4} = 18\tfrac{1}{2},\ \text{1st}, \\ x+8 = 26\tfrac{1}{2},\ \text{2d}, \\ 2x+8 = 45,\ \text{3d}. \end{array}\right\} \text{Ans.}$$

12. Let $x =$ length of the body;
then $\frac{x}{2}+9 =$ " " " tail,
and the body or $x =$ tail or $\frac{x}{2}+9+$ the head or 9;
that is, $x = \frac{x}{2}+9+9$
Cl. of fr. $2x = x+36$,
tr. $2x-x=36$ or $x = 36$, body;
$\frac{x}{2}+9 = 27$, tail;
9, head;
72 in. or 6 ft. whole length.
Ans.

13. Let $x =$ sum divided.
$\frac{x}{4}+150 =$ A's share;
$\frac{x}{3}+300 =$ B's "
$\frac{x}{5}+60 =$ C's "
then $\frac{x}{4}+150+\frac{x}{3}+300+\frac{x}{5}+60 = x$.
Cl. of fr. $15x+9000+20x+18000+12x+3600 = 60x$,
trans. $47x-60x = -30600$,
$-13x = -30600$.
$x = \frac{-30600}{-13} = \$2353\frac{11}{13}$. Ans.

14. Let $x =$ distance.

$\frac{x}{3} = \frac{1}{3}$ distance.

$x - \frac{x}{3} = \frac{2x}{3}$ 1st remainder;

$\frac{1}{3}$ of $\frac{2x}{3} = \frac{2x}{9}$ 2d remainder;

then $\frac{x}{3} + \frac{2x}{9} + 36 = x$, or the whole distance.

Clear of fractions, $3x + 2x + 324 = 9x$,

transpose, $-4x = -324$.

$$x = \frac{-324}{-4} = 81 \text{ miles.} \quad \text{Ans.}$$

15. Let $x =$ whole distance.

$\frac{x}{5} + 8 =$ 1st day's journey.

$x - \left(\frac{x}{5} + 8\right) =$ dist. that remained $= \frac{5x}{5} - \frac{x}{5} - 8 = \frac{4x}{5} - 8.$

$\frac{1}{5}\left(\frac{4x}{5} - 8\right) + 15 =$ 2d day's journey $= \frac{4x}{25} - \frac{8}{5} + 15 = \frac{4x}{25} - \frac{8}{5} + \frac{75}{5} = \frac{4x}{25} + \frac{67}{5}.$

$\left(\frac{4x}{5} - 8\right) - \left(\frac{4x}{25} + \frac{67}{5}\right) =$ distance after 2d day $= \frac{20x}{25} - \frac{40}{5} - \frac{4x}{25} - \frac{67}{5} = \frac{16x}{25} - \frac{107}{5}.$

$\frac{1}{3}\left(\frac{16x}{25} - \frac{107}{5}\right) + 12 =$ 3d day's journey $= \frac{16x}{75} - \frac{107}{15} + 12 = \frac{16x}{75} - \frac{107}{15} + \frac{180}{15} = \frac{16x}{75} + \frac{73}{15}.$

$\left(\frac{16x}{25} - \frac{107}{5}\right) - \left(\frac{16x}{75} + \frac{73}{15}\right) =$ dist. after 3d day $= \frac{48x}{75} - \frac{321}{15} - \frac{16x}{75} - \frac{73}{15} = \frac{32x}{75} - \frac{394}{15}.$

As he finished the journey by travelling 35 miles on the fourth day, this must have been what remained after the third day, or

$$\frac{32x}{75} - \frac{394}{15} = 35.$$

Clear of fractions, $32x - 1970 = 2625$,

transpose, $32x = 2625 + 1970 = 4595$.

$$x = \frac{4595}{32} = 143\tfrac{19}{32} \text{ miles. Ans.}$$

16. Let $x =$ B's share;

$2x =$ A's "

then $2x - 300 = x + \frac{x}{2}$.

Clear of fractions, $4x - 600 = 2x + x$,

transpose, $4x - 3x = 600$.

$$\left.\begin{array}{r} x = 600, \text{ B's share,} \\ 2x = 1200, \text{ A's ``} \end{array}\right\} \text{Ans.}$$

17. Let $x =$ rate.

In 6 hours the merchant vessel would sail 36 miles;

add 15 miles' start $= 51$,

less 1 mile $= 50$ miles to be gained.

In 6 hours at x miles an hour the war vessel would sail $6x$ miles.

$\therefore 6x = 50$.

$x = 8\frac{1}{3}$ miles per hour. Ans.

18. Let $x =$ time of latter;

$x + \frac{1}{2} =$ " " former;

then $mx = h(x + \frac{1}{2})$, or $mx = hx + \frac{1}{2}h$.

Clear of fractions, $2mx = 2hx + h$,

transpose, $2mx - 2hx = h$, or $2(m - h)x = h$.

$$x = \frac{h}{2(m-h)}. \quad \text{Ans.}$$

19. Let $x =$ 1st number;

$x + 9 =$ 2d "

then $(x + 9)^2 - x^2 = 351$.

Clear of fractions,

$x^2 + 18x + 81 - x^2 = 351$, or $18x + 81 = 351$;

transpose, $18x = 351 - 81 = 270$.

$$\left.\begin{array}{r} x = 15, \\ x + 9 = 24. \end{array}\right\} \text{Ans.}$$

20. Let $x =$ number of good horses;
$25 - x =$ " " poor horses.

$$80\,(25 - x) + 130\,x = 2500;$$

or $2000 - 80\,x + 130\,x = 2500.$

Transpose, $-80\,x + 130\,x = 2500 - 2000;$

$$50\,x = 500;$$

$$\left.\begin{aligned} x &= 10, \\ 25 - x &= 15. \end{aligned}\right\} \text{ Ans.}$$

21. Let $x =$ wife's age;
$x + 5 =$ man's age;
$x + 15 =$ wife's age fifteen years hence;
$x + 20 =$ man's " " " "

then $x + 15 + x + 20 = 3\,x.$

Transpose, $2\,x - 3\,x = -35;$

$$-x = -35;$$

$$\left.\begin{aligned} x &= 35, \text{ w.}, \\ x + 5 &= 40, \text{ m.} \end{aligned}\right\} \text{ Ans.}$$

22. Let $x =$ number of hours going;
$8 - x =$ " " " returning;

then $6\,x = 4\,(8 - x)$, or $6\,x = 32 - 4\,x.$

Transpose, $10\,x = 32;$

$x = 3\frac{1}{5}$ hours, or $19\frac{1}{5}$ miles. Ans.

23. $\dfrac{1}{x} = \dfrac{1}{12} + \dfrac{1}{15}.$

Clear of fractions, $60 = 5\,x + 4\,x;$

$$9\,x = 60;$$

$x = 6\frac{6}{9} = 6\frac{2}{3}$ days. Ans.

24. Let $x =$ time required;

$\dfrac{1}{x} =$ part filled in one minute;

$\dfrac{1}{12} =$ " " " " " by both;

$\dfrac{1}{20} =$ " " " " " by one;

then $\dfrac{1}{12} - \dfrac{1}{20} = \dfrac{1}{x}.$

Clear of fractions, $5\,x - 3\,x = 60;$

$$2\,x = 60;$$

$x = 30.$ Ans.

25. Let $x =$ time of first;

$\frac{1}{x} =$ part filled by first in one hour;

$\frac{2}{x} =$ " " " second in one hour;

$\frac{1}{x} + \frac{2}{x} = \frac{3}{x} =$ " " " third in one hour;

then $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 1,$

$1 + 2 + 3 = x.$

$$\left.\begin{aligned} x &= 6, \text{ time of 1st;} \\ \frac{x}{2} &= 3, \text{ " " 2d;} \\ \frac{x}{3} &= 2, \text{ " " 3d.} \end{aligned}\right\} \text{Ans.}$$

26. Let $x =$ number bought;

5 for 2 cts. $= \frac{2}{5}$ ct. each;

x at $\frac{2}{5}$ ct. each $= \frac{2x}{5} =$ the cost;

2 for 1 ct. $= \frac{1}{2}$ ct. each;

3 " " $= \frac{1}{3}$ " "

$\frac{x}{2}$ at $\frac{1}{2} = \frac{x}{4}$. $\frac{x}{2}$ at $\frac{1}{3} = \frac{x}{6}$;

then $\frac{x}{4} + \frac{x}{6} = \frac{2x}{5} + 50.$

Clear of fractions, $15x + 10x = 24x + 3000;$

transpose, $25x - 24x = 3000;$

$x = 3000.$ Ans.

27. Let $x =$ number of pounds added;

50 lbs. at 90 cts. per lb. $= 4500;$

x lbs. at 60 cts. per lb. $= 60x;$

total amount $= x + 50$ lbs.;

total cost $= 4500 + 60x;$

then $\frac{4500 + 60x}{x + 50} = 70$, the price per lb. of the mixture.

Clear of fractions, $4500 + 60x = 70x + 3500;$

transpose, $60x - 70x = 3500 - 4500;$

$-10x = -1000;$

$x = \frac{-1000}{-10} = 100.$ Ans.

28. Let $x =$ number of days idle;

$40 - x =$ " " " he worked;

$150\,(40 - x) =$ amount he received;

$50\,x =$ amount forfeited;

then $150\,(40 - x) - 50\,x = 5200$ cts.;

$6000 - 150\,x - 50\,x = 5200.$

Transpose, $-200\,x = 5200 - 6000 = -800;$

$$\left.\begin{aligned} x &= \frac{-800}{-200} = 4, \\ 40 - x &= 36. \end{aligned}\right\} \text{ Ans.}$$

29. Let $x =$ amount divided;

the amount paid for services $= 2000$;

then $x - 2000 =$ amount equally divided.

$$\frac{x - 2000}{3} = \frac{x}{4}.$$

Clear of fractions, $4\,x - 8000 = 3\,x;$

transpose, $4\,x - 3\,x = 8000;$

$x = 8000.$ **Ans.**

30. Let $x =$ whole amount;

$\frac{x}{3} + 20 =$ share of first;

$x - \left(\frac{x}{3} + 20\right) =$ what remained $=$

$$\frac{3\,x}{3} - \frac{x}{3} - 20 = \frac{2\,x}{3} - 20;$$

$\frac{1}{3}\left(\frac{2\,x}{3} - 20\right) + 20 =$ share of second $=$

$$\frac{2\,x}{9} - \frac{20}{3} + 20 = \frac{2\,x}{9} - \frac{20}{3} + \frac{60}{3} = \frac{2\,x}{9} + \frac{40}{3};$$

$\frac{2\,x}{3} - 20 - \left(\frac{2\,x}{9} + \frac{40}{3}\right) =$ what remained $=$

$$\frac{6\,x}{9} - \frac{60}{3} - \frac{2\,x}{9} - \frac{40}{3} = \frac{4\,x}{9} - \frac{100}{3};$$

$\frac{1}{3}\left(\frac{4\,x}{9} - \frac{100}{3}\right) + 20 =$ share of third $=$

$$\frac{4\,x}{27} - \frac{100}{9} + 20 = \frac{4\,x}{27} - \frac{100}{9} + \frac{180}{9} = \frac{4\,x}{27} + \frac{80}{9}.$$

As this exhausted the amount $=$

$$\left(\frac{4\,x}{9} - \frac{100}{3}\right) - \left(\frac{4\,x}{27} + \frac{80}{9}\right) = 0 = \frac{4\,x}{9} - \frac{100}{3} - \frac{4\,x}{27} - \frac{80}{9} = 0.$$

Clear of fractions, $12\,x - 900 - 4\,x - 240 = 0;$

transpose, $12\,x - 4\,x = 900 + 240;$

$8\,x = 1140;$

$x = \$142.50.$ **Ans.**

31. Let $x =$ price per sheep of 1st lot;

$x - 2 =$ " " " " 2d "

$\frac{720}{x} =$ number of sheep in 1st lot;

$\frac{480}{x-2} =$ " " " " 2d "

then $$\frac{720}{x} = \frac{480}{x-2}.$$

Clear of fractions, $720x - 1440 = 480x$;

transpose, $720x - 480x = 1440$;

$$240x = 1440;$$

$$\left.\begin{aligned} x &= 6, \\ x - 2 &= 4. \end{aligned}\right\} \text{ Ans.}$$

32. Let $x =$ rate of current;

$9 + x =$ " " boat going down;

$9 - x =$ " " " " up;

then $$9 + x = 2(9 - x);$$

$$9 + x = 18 - 2x;$$

transpose, $x + 2x = 18 - 9$;

$$3x = 9;$$

$x = 3$ miles per hour. Ans.

33. Let $x =$ cost of house;

$12\,000 - x =$ amount remaining;

$\frac{1}{3}(12\,000 - x) = \frac{12\,000 - x}{3}$. $\frac{12\,000 - x}{3}$ at 4 per cent $=$

$$\frac{12\,000 - x}{3} \times \frac{4}{100} = \frac{48\,000 - 4x}{300},$$

$$(12\,000 - x) - \frac{12\,000 - x}{3} = \frac{36\,000 - 3x}{3} - \frac{12\,000 - x}{3} = \frac{24\,000 - 2x}{3}, \text{ 2d rem.}$$

$\frac{24\,000 - 2x}{3}$ at 5 per cent $=$

$$\frac{24\,000 - 2x}{3} \text{ at } \frac{5}{100} = \frac{120\,000 - 10x}{300};$$

then $$\frac{48\,000 - 4x}{300} + \frac{120\,000 - 10x}{300} = 392.$$

Clear of fractions, $48\,000 - 4x + 120\,000 - 10x = 117\,600$;

transpose, $-14x = 117\,600 - 48\,000 - 120\,000$;

$$-14x = -50\,400;$$

$$x = \frac{-50\,400}{-14} = \$3600. \text{ Ans.}$$

34. Let $x =$ the income;

$x - 3600 =$ amount taxed at 3 per cent;

$\frac{(x - 3600)3}{100} =$ tax on above amount;

$3000 =$ amount taxed at 2 per cent;

$\$60 =$ tax on above amount;

$\frac{2x}{100} =$ tax on income at 2 per cent;

then $$\frac{(x - 3600)3}{100} + 60 = \frac{2x}{100} + 200.$$

Clear of fractions, $3x - 10800 + 6000 = 2x + 20000$;

transpose, $3x - 2x = +20000 + 10800 - 6000$;

$x = 24800$. Ans.

35. At 3 o'clock the minute-hand is 15 minutes behind the hour-hand. It therefore must gain 20 minute-spaces.

Let $x =$ number of minutes past 3 o'clock;

then $x - 20 =$ number of spaces hour-hand moves; but the minute-hand goes 12 spaces while the hour-hand goes one;

$\therefore 12x - 240 = x$;

transpose, $12x - x = 240$;

$11x = 240$;

$x = 21\frac{9}{11}$ min.,

or the time $= 3$ hrs. $21\frac{9}{11}$ min. Ans.

36. Let $x =$ capacity of dipper;

$x =$ gals. of brandy in first, after changing;

$\frac{x}{a} =$ amount of brandy per gal. of mixture;

$b - x =$ gals. of brandy in second, after changing;

$\frac{b - x}{b} =$ amount of brandy per gal. of mixture;

$$\therefore \frac{x}{a} = \frac{b - x}{b}.$$

Clear of fractions, $bx = ab - ax$;

transpose, $= bx + ax = ab$ or $x(b + a) = ab$;

$$x = \frac{ab}{a + b}. \quad \text{Ans.}$$

37. Let x be the quantity to which the first part added to a, the second diminished by a, etc., must all be equal.

Now if the first part must be increased by a to equal x, then it will equal $x - a$, and likewise

$x + a = 2d, \frac{x}{a} = 3d,$ and $ax = 4th;$

then $x - a + x + a + \frac{x}{a} + ax = m.$

Clear of fractions, $ax - a^2 + ax + a^2 + x + a^2x = am,$
or $a^2x + 2\,ax + x = am;$
$(a^2 + 2\,a + 1)\,x = am;$

$$x = \frac{am}{a^2 + 2\,a + 1}.$$

Now this quantity added to a, diminished by a, etc., will equal the given quantities;

that is, $\frac{am}{1 + 2\,a + a^2} + a, - a, \times a,$ and $\div a =$
the four parts. Ans.

38. Let $x =$ share of youngest;
$x + n =$ " " next younger;
$x + 2\,n =$ " " " "
$x + 3\,n =$ " " " "
$x + 4\,n =$ " " " "
then $x + x + n + x + 2n + x + 3\,n + x + 4\,n = a,$
or $5\,x + 10\,n = a;$
transpose, $5\,x = a - 10\,n,$

$$x = \frac{a - 10\,n}{5}, \text{ 1st;}$$

$$x + n = \frac{a - 10\,n}{5} + n = \frac{a - 10\,n + 5\,n}{5} = \frac{a - 5\,n}{5}, \text{ 2d;}$$

$$x + 2\,n = \frac{a - 5\,n}{5} + n = \frac{a}{5}, \text{ 3d;}$$

$$x + 3\,n = \frac{a}{5} + n = \frac{a + 5\,n}{5}, \text{ 4th.}$$

39. Let $x =$ number of hours before meeting;
$x + 4 =$ " " " first must travel;
at 10 miles per hour the second will travel $10\,x$ miles;
at 8 " " " " first " " $8\,(x + 4);$
since they are to travel the same distance,

$$10\,x = 8\,(x + 4) = 8\,x + 32;$$

transpose, $10\,x - 8\,x = 32;$

$$2\,x = 32;$$
$$x = 16, \text{ the time;}$$
$$10\,x = 160 \text{ miles, the distance.}$$

40. $x =$ the intervals of time;
$7x =$ " distance A goes;
$5x =$ " " B "

But A must go one mile more than B in order to overtake him.

$$\therefore 7x = 5x + 1;$$
$$2x = 1;$$
$$x = \tfrac{1}{2} \text{ hour.}$$

In this half-hour A has made $2\tfrac{1}{2}$ circuits, and B $3\tfrac{1}{2}$. Therefore they overtake each other $\tfrac{1}{2}$ a circuit farther ahead each time, and the points of overtaking will be *two* in number.

41. $8x = 5x + 1;$
$3x = 1;$
$x = \tfrac{1}{3}$ hour.

In this third of an hour A will make $2\tfrac{2}{3}$ circuits and B $1\tfrac{2}{3}$. Therefore their points of meeting will be $\tfrac{1}{3}$ of a circuit apart, and the points will be *three* in number.

42. Let x be the required number of seconds. In x seconds the faster will have made $\frac{x}{30}$ rounds, and the slower $\frac{x}{35}$ rounds.

The horses first come together when the swifter has made one round more than the slower. Hence

$$\frac{x}{30} - \frac{x}{35} = 1.$$
$$\times\ 210;\ 7x - 6x = 210,$$
$$\text{or } x = 210.$$
$$\text{Time} = 210 \text{ sec.} = 3 \text{ min. } 30 \text{ sec.}$$

43. Reasoning as in the last problem, if x be the required interval, one planet will have made $\frac{x}{T}$, and the other $\frac{x}{T'}$ revolutions. Hence

$$\frac{x}{T'} - \frac{x}{T} = 1,$$

which gives $x = \frac{TT'}{T - T'}$, the required interval.

§ 138.

1. $$\begin{aligned} \times (1) \text{ by } 2 &= 6x - 4y = 66; \\ \times (2) \text{ by } 3 &= 6x - 9y = 54; \\ (1) - (2) &= 5y = 12; \\ y &= \frac{12}{5}. \end{aligned}$$

$$\text{Sub. in (1) } 6x - \frac{48}{5} = 66;$$

$$\text{trans., } 6x = \frac{48}{5} + 66 = \frac{48}{5} + \frac{330}{5} = \frac{378}{5};$$

$$\left.\begin{aligned} x &= \frac{63}{5}, \\ y &= \frac{12}{5}. \end{aligned}\right\} \text{ Ans.}$$

2. $$\begin{aligned} \times (1) \text{ by } 2 &= 6x - 10y = 26; \\ \times (2) \text{ by } 3 &= 6x + 21y = 243; \\ (2) - (1) &= 31y = 217; \\ y &= 7. \\ \text{Sub. in (1) } 6x - 70 &= 26; \\ \text{transpose, } 6x &= 96; \end{aligned}$$

$$\left.\begin{aligned} x &= 16, \\ y &= 7. \end{aligned}\right\} \text{ Ans.}$$

3. From (1) $6y = a - 7x$.
Sub. in (2) $6x + (a - 7x) = b;$

$$\begin{aligned} 6x + a - 7x &= b. \\ \text{Transpose, } 6x - 7x &= b - a; \\ -x &= b - a; \\ x &= a - b. \end{aligned}$$

Sub. in (1) $7x + 6y = a;$

$$\begin{aligned} 7(a - b) + 6y &= a; \\ 7a - 7b + 6y &= a. \end{aligned}$$

Transpose, $6y = a - 7a + 7b = -6a + 7b;$

$$\left.\begin{aligned} y &= \frac{7b - 6a}{6} = \frac{7b}{6} - a, \\ x &= a - b. \end{aligned}\right\} \text{ Ans.}$$

4. From (1) $2x = m - 3y$.

Sub. in (2) $(m - 3y) - 3y = n$;

$$m - 3y - 3y = n.$$

Transpose, $-6y = n - m$;

$$y = \frac{-m+n}{-6} = \frac{m-n}{6}.$$

Sub. in (1) $2x + 3\left(\frac{m-n}{6}\right) = m$;

$$2x + \frac{3m - 3n}{6} = m.$$

Clear of fractions, $= 12x + 3m - 3n = 6m$;

trans., $12x = 6m - 3m + 3n = 3m + 3n = 3(m+n)$;

$$\left.\begin{aligned} x &= \frac{3(m+n)}{12} = \frac{m+n}{4}, \\ y &= \frac{m-n}{6}. \end{aligned}\right\} \text{ Ans.}$$

5. $(1) + (2) = 2ax = p + q$;

$$x = \frac{p+q}{2a}.$$

$(1) - (2) = 2by = p - q$;

$$\left.\begin{aligned} y &= \frac{p-q}{2b}, \\ x &= \frac{p+q}{2a}. \end{aligned}\right\} \text{ Ans.}$$

6. $(1) + (2) = \frac{2x}{6} = 28$.

Clear of fractions, $2x = 168$;

$$x = 84;$$

$(1) - (2) = \frac{2y}{7} = 24$.

Clear of fractions, $2y = 168$;

$$\left.\begin{aligned} y &= 84, \\ x &= 84. \end{aligned}\right\} \text{ Ans.}$$

7. Clear of fractions, $(1) = 5x + 4y = 360$;

$$(2) = x + 4y = 232;$$

$$(1) - (2) = 4x = 128;$$

$$x = 32.$$

Sub. in (2) $32 + 4y = 232$;

transpose, $4y = 232 - 32 = 200$;

$$\left.\begin{aligned} y &= 50, \\ x &= 32. \end{aligned}\right\} \text{ Ans.}$$

8. $(1)+(2)=\dfrac{2x}{2}=a+b.$

Clear of fractions, $2x=2(a+b);$

$$x=\frac{2(a+b)}{2}=a+b.$$

$(1)-(2)=\dfrac{2y}{3}=a-b.$

Clear of fractions, $2y=3(a-b);$

$$\left.\begin{aligned} y &= \tfrac{3}{2}(a-b),\\ x &= a+b. \end{aligned}\right\}\ \text{Ans.}$$

9. $(1)+(2)=14(x+y)=168;$

$$(3)\quad x+y=12.$$

$(1)-(2)=6(x-y)=36;$

$$(4)\quad x-y=6.$$

$(3)+(4)=\qquad 2x=18;$

$$x=9.$$

$(3)-(4)=\qquad 2y=6;$

$$\left.\begin{aligned} y &= 3,\\ x &= 9. \end{aligned}\right\}\ \text{Ans.}$$

10. $(1)+(2)=2(x+y)=24;$

$$(3)\quad x+y=12.$$

$(1)-(2)=2(x-y)=4;$

$$(4)\quad x-y=2.$$

$(3)+(4)=\qquad 2x=14;$

$$x=7.$$

$(3)-(4)=\qquad 2y=10;$

$$\left.\begin{aligned} y &= 5,\\ x &= 7. \end{aligned}\right\}\ \text{Ans.}$$

11. $(1)+(2)=\dfrac{2}{x}=\dfrac{6}{12}.$

Clear of fractions, $24=6x;$

$$x=4.$$

$(1)-(2)=\dfrac{2}{y}=\dfrac{4}{12}.$

Clear of fractions, $24=4y;$

$$\left.\begin{aligned} y &= 6,\\ x &= 4. \end{aligned}\right\}\ \text{Ans.}$$

13. $(1) + (2) = \frac{4}{x} = \frac{6}{12}$.

Clear of fractions, $48 = 6x$;

$x = 8$.

$(1) - (2) = \frac{6}{y} = \frac{8}{12}$.

Clear of fractions, $72 = 8y$;

$\left.\begin{array}{l} y = 9, \\ x = 8. \end{array}\right\}$ Ans.

14. $\times$ (1) by $2 = \frac{2}{x} + \frac{4}{y} = \frac{10}{12}$ (3);

$(2) - (3) = \frac{5}{y} = \frac{10}{12} - \frac{5}{24} = \frac{15}{24}$;

$\frac{1}{y} = \frac{3}{24}$;

$y = \frac{24}{3} = 8$.

$\times$ (2) by $2 = \frac{4}{x} - \frac{2}{y} = \frac{5}{12}$ (4);

$(1) + (4) = \frac{5}{x} = \frac{10}{12}$;

$\frac{1}{x} = \frac{2}{12}$;

$\left.\begin{array}{l} x = \frac{12}{2} = 6, \\ y = 8. \end{array}\right\}$ Ans.

15. $\times$ (1) by $3 = \frac{15}{x} - \frac{9}{y} = -\frac{3}{6}$ (3);

$\times$ (2) by $5 = \frac{15}{x} - \frac{5}{y} = \frac{5}{30} = \frac{1}{6}$ (4);

$(4) - (3) = \frac{4}{y} = \frac{4}{6}$ or $\frac{1}{y} = \frac{1}{6}$;

$6 = y$.

Sub. in (2), $\frac{3}{x} - \frac{1}{6} = \frac{1}{30}$;

transpose, $\frac{3}{x} = \frac{1}{30} + \frac{1}{6} = \frac{6}{30}$;

$\frac{1}{x} = \frac{2}{30}$;

$30 = 2x$;

$\left.\begin{array}{l} x = 15, \\ y = 6. \end{array}\right\}$ Ans.

16. $\times$ (1) by 3 $= \dfrac{15}{x+1} - \dfrac{9}{y-1} = -\dfrac{3}{6}$ (3);

$\times$ (2) by 5 $= \dfrac{15}{x+1} - \dfrac{5}{y-1} = \dfrac{5}{30} = \dfrac{1}{6}$ (4);

(4) $-$ (3) $= \dfrac{4}{y-1} = \dfrac{4}{6}$ or $\dfrac{1}{y-1} = \dfrac{1}{6}$.

Clear of fractions, $6 = y - 1$;

transpose, $7 = y$.

Sub. in (2) $= \dfrac{3}{x+1} - \dfrac{1}{7-1} = \dfrac{1}{30} = \dfrac{3}{x+1} - \dfrac{1}{6} = \dfrac{1}{30}$;

transpose, $\dfrac{3}{x+1} = \dfrac{1}{30} + \dfrac{1}{6} = \dfrac{6}{30}$;

$$\frac{1}{x+1} = \frac{2}{30}.$$

Clear of fractions, $30 = 2x + 2$;

transpose, $28 = 2x$;

$$\left.\begin{aligned} x &= 14, \\ y &= 7. \end{aligned}\right\} \text{ Ans.}$$

17. (1) + (2) $= \dfrac{4}{x+2} = \dfrac{6}{12}$.

Clear of fr., $48 = 6x + 12$;

transpose, $36 = 6x$;

$6 = x$.

(1) $-$ (2) $= \dfrac{6}{y-3} = \dfrac{8}{12}$.

Clear of fr., $72 = 8y - 24$;

transpose, $96 = 8y$;

$$\left.\begin{aligned} 12 &= y, \\ 6 &= x. \end{aligned}\right\} \text{ Ans.}$$

18. (1) + (2) $= \dfrac{2a}{x} = c + d$.

Clear of fr., $2a = x(c+d)$;

$$\frac{2a}{c+d} = x.$$

(1) $-$ (2) $= \dfrac{2b}{y} = c - d$.

Clear of fr., $2b = y(c-d)$;

$$\frac{2b}{c-d} = y;$$

$$\left.\begin{aligned} x &= \frac{2a}{c+d}, \\ y &= \frac{2b}{c-d}. \end{aligned}\right\} \text{ Ans.}$$

19. Clear of fractions, $x + y = 2x - 2y$;

$$
\begin{aligned}
\text{transpose, } 3y &= x; \\
2x + 3y &= 18. \\
\text{Sub. } x, \quad 6y + 3y &= 18; \\
9y &= 18; \\
\left.\begin{aligned} y &= 2, \\ x = 3y &= 6. \end{aligned}\right\} &\quad \text{Ans.}
\end{aligned}
$$

20. Clear of fr., $\left.\begin{matrix}(1)\\(2)\end{matrix}\right\}$ $ax - bx + ay + by = 2a^3 - 2ab^2$ (3).

" " $x - y = 4ab$ (4);

$$
\begin{aligned}
x &= 4ab + y. \\
\text{Sub. in (3), } 4a^2b + ay - 4ab^2 - by + ay + by &= 2a^3 - 2ab^2; \\
2ay &= 2a^3 - 2ab^2 + 4ab^2 - 4a^2b; \\
2ay &= 2a^3 - 4a^2b + 2ab^2 = 2a(a^2 - 2ab + b^2); \\
\left.\begin{aligned} y &= \frac{2a(a^2 - 2ab + b^2)}{2a} = a^2 - 2ab + b^2, \\ x &= 4ab + a^2 - 2ab + b^2 = a^2 + 2ab + b^2. \end{aligned}\right\} &\quad \text{Ans.}
\end{aligned}
$$

§ 140.

2. $(1) + (2) + (3) + (4) = 4x_1 = 108$;

$$
\begin{aligned}
x_1 &= 27. \\
(1) + (3) = 2x_1 + 2x_3 &= 70; \\
x_1 + x_3 &= 35. \\
\text{Sub. } x_1, \quad 27 + x_3 &= 35; \\
\text{transpose, } \quad x_3 &= 8. \\
(1) + (4) = 2x_1 + 2x_4 &= 68; \\
x_1 + x_4 &= 34. \\
\text{Sub. } x_1, \quad 27 + x_4 &= 34; \\
\text{transpose, } \quad x_4 &= 7. \\
\text{Sub. in (1), } \quad 27 + x_2 + 8 + 7 &= 64; \\
\text{transpose, } \quad x_2 &= 64 - 42; \\
x_2 &= 22.
\end{aligned}
$$

3.

$$
\begin{aligned}
(1) - (2) &= 3y + 4z = 1 \text{ (4)}; \\
\times (1) \text{ by } 5 &= 10x + 25y + 15z = 65 \text{ (5)}; \\
\times (3) \text{ by } 2 &= 10x + 10y - 4z = 58 \text{ (6)}; \\
(5) - (6) &= 15y + 19z = 7 \text{ (7)}; \\
\times (4) \text{ by } 5 &= 15y + 20z = 5 \text{ (8)}; \\
(8) - (7) &= z = -2.
\end{aligned}
$$

$$
\begin{aligned}
\text{Sub. in (4), } 3y - 8 &= 1; \\
3y &= 9; \\
y &= 3. \\
\text{Sub. in (1), } 2x + 15 - 6 &= 13; \\
2x &= 4; \\
x &= 2.
\end{aligned}
$$

4. $\times$ (3) by 3 $= 3x + 21z - 18y = 99$ (5);
(5) $-$ (2) $= 21z - 19y + 4u = 90$ (6);
$\times$ (1) by 7 $= 21z + 14u - 35y = 126$ (7);
(7) $-$ (6) $= 10u - 16y = 36$ (8);
$\times$ (3) by 2 $= 2x + 14z - 12y = 66$ (9);
(4) $+$ (9) $= 19z - 20y + 2u = 81$ (10);
$\times$ (1) by 19 $= 57z + 38u - 95y = 342$ (11);
$\times$ (10) by 3 $= 57z - 60y + 6u = 243$ (12);
(11) $-$ (12) $= 32u - 35y = 99$ (13);
$\times$ (13) by 5 $= 160u - 175y = 495$ (14);
$\times$ (8) by 16 $= 160u - 256y = 576$ (15);
(15) $-$ (14) $= -81y = 81$;
$y = -1.$
Sub. in (8), $10u + 16 = 36$;
transpose, $10u = 20$;
$u = 2.$
Sub. in (1), $3z + 4 + 5 = 18$;
transpose, $3z = 9$;
$z = 3.$
Sub. in (2), $3x - 1 - 8 = 9$;
transpose, $3x = 18$;
$x = 6.$

5. (1) $+$ (2) $+$ (3) $+$ (4) $=$
$3x + 3y + 3z + 3u = a + b + c + d$;
$\div 3 = x + y + z + u = \dfrac{a+b+c+d}{3}$ (5);

$$\left.\begin{aligned}
(5)-(1) &= u = \frac{a+b+c+d}{3} - a = \frac{b+c+d-2a}{3};\\
(5)-(2) &= x = \frac{a+b+c+d}{3} - b = \frac{a+c+d-2b}{3};\\
(5)-(3) &= y = \frac{a+b+c+d}{3} - c = \frac{a+b+d-2c}{3};\\
(5)-(4) &= z = \frac{a+b+c+d}{3} - d = \frac{a+b+c-2d}{3}.
\end{aligned}\right\}\text{Ans.}$$

6. (1) $+$ (2) $+$ (3) $= \dfrac{2}{x} = m + n + p.$
Clear of fractions, $2 = x(m + n + p)$;
$$\frac{2}{m+n+p} = x;$$
(2) $+$ (3) $= \dfrac{1}{y} + \dfrac{1}{x} = n + p$ (4);

$$(1) - (4) = -\frac{2}{y} = m - (n + p) = m - n - p;$$

$$y = \frac{2}{n + p - m};$$

$$(1) + (2) = \frac{1}{x} - \frac{1}{z} = m + n \ (5);$$

$$(5) - (3) = -\frac{2}{z} = m + n - p;$$

$$z = \frac{2}{p - m - n}.$$

§ 140.—Problems.

1. Let x = price of horse A;
y = " " " B;
then $x + 75 = 2y$ (1);
$y + 75 = x$ (2).
Sub. value of $x = y + 75$ in (1) $y + 75 + 75 = 2y$;
transpose, $-y = -150$;
$y = 150$, A,
$x = y + 75 = 225$, B. } Ans.

2d is solved.

3. Let x be the 1st digit;
y " 2d "
then $10x + y = 6(x + y) = 6x + 6y$ (1);
$10x + y - 9 = 10y + x$ (2);
transpose, (1) $= 4x - 5y = 0$ (3);
" (2) $= 9x - 9y = 9$ (4);
$\div$ (4) by $9 = x - y = 1$ (5);
$\times$ (5) by $4 = 4x - 4y = 4$ (6);
(6) $-$ (3) $= y = 4$.
From (5), $x = 1 + 4 = 5$. } Ans. Or number = 54.

4. Let x be the 1st digit;
y " 2d "
then $10x + y = 6(x + y + 1) = 6x + 6y + 6$ (1);
$10x + y - 18 = 10y + x$ (2);
trans., (1) $= 4x - 5y = 6$ (3);
" (2) $= 9x - 9y = 18$ (4);
$\div$ (4) by $9 = x - y = 2$ (5);
$\times$ (5) by $4 = 4x - 4y = 8$ (6);
(6) $-$ (3) $= y = 2$.
From (5), $x = 2 + 2 = 4$. } No. = 42. Ans.

5. Let x be the 1st digit;
y " 2d "
then $10x + y = 5(x + y) + 2 = 5x + 5y + 2$ (1);
$10x + y + 9 = 10y + x$ (2);
trans., (1) $= 5x - 4y = 2$ (3);
" (2) $= 9x - 9y = -9$ (4);
$\div$ (4) by 9 $= x - y = -1$ (5);
$\times$ (5) by 4 $= 4x - 4y = -4$ (6);
(3) $-$ (6) $= x = 6$.
From (5), $-y = -1 - 6 = -7$;
$y = 7$.
No. = 67. Ans.

6. Let x be the 1st digit;
y " 2d "
then $10x + y = 9(x + y) = 9x + 9y$ (1);
$10x + y = 11(x - y) + 4 = 11x - 11y + 4$ (2);
trans., (1) $= x - 8y = 0$ (3);
" (2) $= -x + 12y = 4$ (4);
(3) + (4) $= 4y = 4$;
$y = 1$.
Sub. in (3), $x = 8$. No. = 81. Ans.

7. Let x be the numerator;
y " denominator;
then $\dfrac{x+2}{y} = \dfrac{2}{3}$, or, cl. of fractions, $3x + 6 = 2y$ (1);
$\dfrac{x}{y+4} = \dfrac{4}{7}$, " " " " $7x = 4y + 16$ (2);
transpose, (1) $= 3x - 2y = -6$ (3);
" (2) $= 7x - 4y = 16$ (4);
$\times$ (3) by 2 $= 6x - 4y = -12$ (5);
(4) $-$ (5) $= x = 28$.
From (3), $-2y = -6 - 3x = -6 - 84 = -90$;
$y = 45$.

$\dfrac{28}{45}$ = fraction. Ans.

8. Let x = A's;
y = B's;
then $x - 50 = \dfrac{y}{2}$, or $2x - 100 = y$ (1);
$y - 54 = \dfrac{x - 50}{2}$, or $2y - 108 = x - 50$ (2);

transpose, (1) $= 2x - y = 100$ (3);
" (2) $= -x + 2y = 58$ (4);
(4) + (3) $= x + y = 158$ (5);
(5) + (4) $= 3y = 216$.

$$\left.\begin{aligned} y &= 72, \text{ B}; \\ \text{from (3), } 2x &= 172; \\ x &= 86, \text{ A}. \end{aligned}\right\} \text{Ans.}$$

9. Let x = number of good ones;
y = " " bad "
then $x + y = 42$ (1);
$3x + 2y = 100$ (2);
$\times$ (1) by $2 = 2x + 2y = 84$ (3);

$$\left.\begin{aligned} (2) - (3) &= x = 16; \\ \text{from (1), } y &= 42 - 16 = 26. \end{aligned}\right\} \text{Ans.}$$

10. Let x be the numerator;
y " denominator;

then $\dfrac{x}{y+13} = \dfrac{1}{2}$, or $2x = y + 13$, or tr., $2x - y = 13$ (1);

$\dfrac{x-4}{y} = \dfrac{2}{3}$, or $3x - 12 = 2y$, or tr., $3x - 2y = 12$ (2);

(2) − (1) $= x - y = -1$ (3);
(3) − (1) $= -x = -14$, $x = 14$;
from (1), $2x - y = 13$;
$2(14) - y = 13$;
$-y = +13 - 28 = -15$;
$y = 15$.

$\therefore \dfrac{14}{15}$, the fraction. Ans.

11. Let x be the numerator;
y " denominator;

then $\dfrac{x+2}{y} = \dfrac{2}{3}$, or $3x + 6 = 2y$ (1);

and $\dfrac{x}{y+3} = \dfrac{1}{3}$, or $3x = y + 3$ (2);

(1) − (2) $= 6 = y - 3$;
transpose, $9 = y$.
Sub. in (2), $3x = 9 + 3$;
$3x = 12$;
$x = 4$.

$\therefore \dfrac{4}{9}$, the fraction. Ans.

12. Let $x =$ number of turkeys;
$y =$ " " chickens;
then $75x + 32y = 1400$ (1);
$100x + 48y = 2000$ (2);
$\div$ (2) by $4 = 25x + 12y = 500$ (3);
$\times$ (3) by $3 = 75x + 36y = 1500$ (4);
(4) − (1) $= 4y = 100$;
$y = 25$;
from (3), $25x = 500 - 12y = 500 - 300 = 200$.
$\left.\begin{array}{l} x = 8, \\ y = 25. \end{array}\right\}$ Ans.

13. Let $x =$ number of pears;
$y =$ " " apples;
$x - 7 =$ " " good pears;
$y - 11 =$ " " " apples;
then $y + 2x = 170$ (1);
(2) $2(y - 11) + 3(x - 7) = 260$, or $2y - 22 + 3x$
$- 21 = 260$, or $2y + 3x = 303$;
$\times$ (1) by $2 = 2y + 4x = 340$ (3);
(3) − (2) $= x = 37$;
from (1), $y = 170 - 2x = 170 - 74 = 96.$ } Ans.

14. Let $x =$ Mr. S's age when married;
$y =$ Mrs. " " " "
then $x = y + \frac{y}{3}$, or $3x = 3y + y$, or $3x - 4y = 0$ (1);
$x + 12 = y + 12 + \frac{y + 12}{5}$, or $5x + 60 = 5y + 60 + y$
$+ 12$, or $5x - 6y = 12$ (2);
$\times$ (1) by $3 = 9x - 12y = 0$ (3);
$\times$ (2) by $2 = 10x - 12y = 24$ (4);
(4) − (3) $= x = 24$;
from (1), $3x = 4y$;
$72 = 4y$;
$\left.\begin{array}{l} 18 = y, \\ 24 = x. \end{array}\right\}$ Ans.

15. Let $x =$ time required by A;
$y =$ " " " B;
then $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$ (1);
$\frac{1}{x} = \frac{1}{y} + \frac{1}{18}$ (2);

Sub. $\frac{1}{x}$ in (1) $= \frac{1}{y} + \frac{1}{18} + \frac{1}{y} = \frac{1}{6}$;

transpose, $\frac{2}{y} = -\frac{1}{18} + \frac{1}{6} = -\frac{1}{18} + \frac{3}{18} = \frac{1}{9}$;

$\frac{1}{y} = \frac{1}{18}$;

$y = 18.$

Sub. in (1), $\frac{1}{x} + \frac{1}{18} = \frac{1}{6}$;

transpose. $\frac{1}{x} = \frac{1}{6} - \frac{1}{18} = \frac{3}{18} - \frac{1}{18} = \frac{1}{9}$.

$\left.\begin{array}{l} x = \ \ 9,\text{ A's time;} \\ y = 18,\text{ B's time.} \end{array}\right\}$ Ans.

16. Let x = husband's age;
y = wife's "

$\frac{x+6}{y-6} = 2$, or $x + 6 = 2y - 12$; hence $x - 2y = -18$ (1);

$\frac{y+12}{x-21} = 5$, or $y + 12 = 5x - 105$; hence $y - 5x = -117$ (2);

$\times$ (2) by 2 $= 2y - 10x = -234$ (3);
(3) + (1) $-9x = -252$, or $x = 28$.
Sub. in (1), $28 - 2y = -18$, $-2y = -56$, $y = 23$.

17. Let x = 1st age;
y = 2d "
then $x + y = 9(x - y) = 9x - 9y$, or $-8x + 10y = 0$ (1);
and $(x - 7) + (y - 7) = 7[(x - 7) - (y - 7)]$,
or $x + y - 14 = 7x - 7y$, or $-6x + 8y = 14$ (2);
$\times$ (1) by 3 $= -24x + 30y = 0$ (3);
$\times$ (2) by 4 $= -24x + 32y = 56$ (4);
(4) $-$ (3) $= 2y = 56$, $y = 28$.
Sub. in (1), $-8x = -280$;
$x = 35.$

18. Let x = rate of faster train;
y = " " slower "
1 hour 24 minutes = 1.4 hours;
then $1.4x + 1.4y = 98$;
$\times$ by 10, $14x + 14y = 980$ (1);
$3x = 4y$;
$x = \frac{4}{3}y.$

Sub. in (1), $\frac{56}{3}y + 14y = 980.$

Clear of fr., $56y + 42y = 2940;$

$$98y = 2940;$$

$$\left.\begin{aligned} y &= 30 \text{ miles per hour,} \\ x = \tfrac{4}{3}y &= 40 \quad \text{"} \quad \text{"} \quad \text{"} \end{aligned}\right\} \text{Ans.}$$

19. Let x = price of tea;
y = " " coffee;
then $50x + 100y = 6000$ (1);

$50\left(\frac{5}{4}x\right) + 100\left(\frac{4}{3}y\right) = 77$, or $\frac{250x}{4} + \frac{400y}{3} = 7700,$

or, cl. of fractions, $750x + 1600y = 92\,400$ (2);
$\times$ (1) by $15 = 750x + 1500y = 90\,000$ (3);
(2) $-$ (3) $= 100y = 2400;$

$$y = 24;$$

from (1), $50x = 6000 - 100y = 6000 - 2400 = 3600.$

$$\left.\begin{aligned} x &= 72, \\ y &= 24. \end{aligned}\right\} \text{Ans.}$$

20. Let x = price of tea;
y = " " coffee;
then $ax + by = p$ (1);
and $mx + ny = p$ (2);
$\times$ (1) by $m = amx + bmy = mp$ (3);
$\times$ (2) by $a = amx + any = ap$ (4).
(4) $-$ (3) $= any - bmy = ap - mp;$

$$y(an - bm) = p(a - m);$$

$$y = \frac{p(a - m)}{an - bm};$$

$\times$ (1) by $n = anx + bny = np$ (5);
$\times$ (2) by $b = bmx + bny = bp$ (6);
(6) $-$ (5) $= bmx - anx = bp - np;$

$$x(bm - an) = p(b - n);$$

$$x = \frac{p(b - n)}{bm - an}.$$

21. Let x = amount from 1st ingot;
y = " " 2d "

From the 1st, $\frac{x}{2}$ = silver taken;

$$\frac{x}{2} = \text{gold taken.}$$

From the 2d, $\frac{y}{6}$ = silver taken;

$\frac{5y}{6}$ = gold taken.

Hence, total silver $= \frac{x}{2} + \frac{y}{6}$;

total gold $= \frac{x}{2} + \frac{5y}{6}$.

Because the gold is 4 times the silver,

$$\frac{x}{2} + \frac{5y}{6} = 4\left(\frac{x}{2} + \frac{y}{6}\right) = 2x + \frac{2y}{3}.$$

Clear of fractions, $3x + 5y = 12x + 4y$;

or transpose, $y = 9x$ (1).

$x + y = 5$ ounces (2).

Sub. from (1), $x + 9x = 5$;

$10x = 5$;

$x = \frac{1}{2}$;

$y = 5 - x$;

$y = 5 - \frac{1}{2} = 4\frac{1}{2}$.

22. Let x = number of 1st denomination;

y = " " 2d "

$\frac{100}{a}$ = value of each piece of 1st denomination;

$\frac{100}{b}$ = " " " 2d

then $x\frac{100}{a} + y\frac{100}{b} = 100$,

or $\frac{100x}{a} + \frac{100y}{b} = 100$.

Clear of fractions, $100bx + 100ay = 100ab$;

$\div$ 100; $bx + ay = ab$ (1).

Because there are c pieces in all,

$x + y = c$ (2);

(2) $\times a$, $ax + ay = ac$ (3);

(2) $\times b$, $bx + by = bc$ (4).

(1) $-$ (3), $(b - a)x = a(b - c)$;

$$x = \frac{a(b-c)}{b-a}.$$

(1) $-$ (4), $(a - b)y = b(a - c)$;

$$y = \frac{b(a-c)}{a-b}.$$

23. Let $x =$ A's investment;

$x + 1000 =$ B's "

$x + 1500 =$ C's "

$\frac{y}{100} =$ A's rate;

$\frac{y+1}{100} =$ B's "

$\frac{y+2}{100} =$ C's "

$\frac{xy}{100} =$ A's income;

$(x + 1000)\frac{y+1}{100} =$ B's "

$(x + 1500)\frac{y+2}{100} =$ C's "

then $\frac{xy}{100} = (x + 1000)\frac{y+1}{100} - 80.$

Clear of fractions, $xy = (x + 1000)(y + 1) - 8000;$

or $xy = xy + x + 1000\,y + 1000 - 8000;$

transpose, $xy - xy - x - 1000\,y = -7000;$

$x + 1000\,y = 7000$ (1).

Again, $(x + 1000)\frac{y+1}{100} = (x + 1500)\frac{y+2}{100} - 70.$

Cl. of fr., $(x + 1000)(y + 1) = (x + 1500)(y + 2) - 7000;$

or $xy + x + 1000\,y + 1000 = xy + 2\,x + 1500\,y + 3000 - 7000;$

trans., $xy - xy + x - 2\,x + 1000\,y - 1500\,y = -5000;$

or $x + 500\,y = 5000$ (2);

(1) $-$ (2) $= 500\,y = 2000;$

$y = 4;$

$y + 1 = 5;$

$y + 2 = 6;$

then (2) $x = 5000 - 500\,y = 5000 - 2000 = 3000;$

$x + 1000 = 4000;$

$x + 1500 = 4500.$

24. Let x, y, and z represent the gallons in the several casks; then $x + y + z = 344$ (1);

$x - 50 + \frac{y}{3} =$ quantity in 1st after selling 50 gallons and pouring in one third of 2d;

$\frac{2\,y}{3} + \frac{z}{5} =$ quantity in the 2d after adding $\frac{1}{5}$ of 3d;

$\frac{4\,z}{5} =$ quantity remaining in the 3d;

Because the 1st contains 10 gallons more than the 2d,

$$x - 50 + \frac{y}{3} = \frac{2y}{3} + \frac{z}{5} + 10;$$

or, cl. of fr., $15x - 750 + 5y = 10y + 3z + 150;$

or, trans., $15x - 5y - 3z = 900$ (2);

also, $\frac{2y}{3} + \frac{z}{5} = \frac{4z}{5} + 10;$

or, cl. of fractions, $10y + 3z = 12z + 150;$

or, trans., $10y - 9z = 150$ (3);

$$\begin{aligned}
\times (1) \text{ by } 15 &= 15x + 15y + 15z = 5160 \quad (4);\\
(4) - (2) &= 20y + 18z = 4260 \quad (5);\\
\times (3) \text{ by } 2 &= 20y - 18z = 300 \quad (6);\\
(5) - (6) &= 36z = 3960;\\
z &= 110;
\end{aligned}$$

from (3), $10y = 150 + 9z = 150 + 990 = 1140;$

$y = 114;$

from (1), $x = 344 - y - z = 344 - 114 - 110 = 344 - 224 = 120.$

$$\left.\begin{aligned} x &= 120,\\ y &= 114,\\ z &= 110.\end{aligned}\right\} \text{ Ans.}$$

§ 146.

1. Because $(a + b)^2$ must be positive,

$\therefore a^2 + b^2 - 2ab > 0.$

Now, since $x < 1$,

$2abx < 2ab; \quad \therefore a^2 + b^2 - 2abx > 0.$

2. By Ex. 1, $a^2 + b^2 > 2ab;$

$2ab = 2ab;$

Add, $(a + b)^2 > 4ab;$

$\div 4$, $\left(\frac{a + b}{2}\right)^2 > ab.$

3. Transpose, $3x > 18;$

$\therefore \quad x > 6.$

4. Clear of fractions, $12x > 3x + 36;$

transpose, $9x > 36;$

$\therefore \quad x > 4.$

5. Clear of fractions, $21x - 25x > 5x - 45;$

transpose, $-9x > -45;$

$9x < 45;$

$\therefore \quad x < 5.$

6. Transpose, $qx - nx > p - m$;

$$x(q-n) > p - m;$$
$$\therefore \quad x > \frac{p-m}{q-n}.$$

7. Clear of fractions, $xy - y^2 < my - mx$;

transpose, $xy + mx < my + y^2$;

$$x(y+m) < y(m+y); \quad \div m + y;$$
$$\therefore \quad x < y.$$

8. Since $(a-b)^2 + (b-c)^2 + (c-a)^2$ is a sum of squares, it must equal a positive quantity, say x; then

$$a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ac + a^2 = x;$$

or $2a^2 - 2ab + 2b^2 - 2bc + 2c^2 - 2ac = x$;

$\div 2$ and tr., $a^2 + b^2 + c^2 = \frac{x}{2} + ab + bc + ac.$

But $a^2 + b^2 + c^2 = 1$;

$$\therefore \quad \frac{x}{2} - ab + bc + ac = 1;$$

trans., $ab + bc + ac = 1 - \frac{x}{2}$, or < 1.

§ 164.

1. (1) $5 : 3 = 8 : x = 12 : y = 17 : z = 20 : u = 33 : v$.

By Theorem IV.,

$$\left.\begin{aligned}
&\frac{5}{3} = \frac{8}{x} = \frac{12}{y} = \frac{17}{z} = \frac{20}{u} = \frac{33}{v}.\\
&\text{Hence } x = 8 \div \tfrac{5}{3} = \tfrac{24}{5} = 4\tfrac{4}{5};\\
&\qquad y = 12 \div \tfrac{5}{3} = \tfrac{36}{5} = 7\tfrac{1}{5};\\
&\qquad z = 17 \div \tfrac{5}{3} = \tfrac{51}{5} = 10\tfrac{1}{5};\\
&\qquad u = 20 \div \tfrac{5}{3} = \tfrac{60}{5} = 12;\\
&\qquad v = 33 \div \tfrac{5}{3} = \tfrac{99}{5} = 19\tfrac{4}{5}.
\end{aligned}\right\} \text{Ans.}$$

(2) $3 : 5 = 6 : a = 8 : b = 16 : c = 20 : d = 29 : e$.

$$\frac{3}{5} = \frac{6}{a} = \frac{8}{b} = \frac{16}{c} = \frac{20}{d} = \frac{29}{e}.$$

$$\left.\begin{aligned}
&\text{Hence } a = 6 \div \tfrac{3}{5} = \tfrac{30}{3} = 10;\\
&\qquad b = 8 \div \tfrac{3}{5} = \tfrac{40}{3} = 13\tfrac{1}{3};\\
&\qquad c = 16 \div \tfrac{3}{5} = \tfrac{80}{3} = 26\tfrac{2}{3};\\
&\qquad d = 20 \div \tfrac{3}{5} = \tfrac{100}{3} = 33\tfrac{1}{3};\\
&\qquad e = 29 \div \tfrac{3}{5} = \tfrac{145}{3} = 48\tfrac{1}{3};
\end{aligned}\right\} \text{Ans.}$$

3. Let x = 1st part;
y = 2d "
z = 3d "
then $x : 2 = y : 4 = z : 11;$
or $x : y : z = 2 : 4 : 11.$
By Theorem IV.,

$$\frac{x}{2} = \frac{y}{4} = \frac{z}{11} = \frac{x+y+z}{2+4+11} = \frac{102}{17} = 6.$$

$$\left.\begin{aligned} \therefore\ x &= 6 \div \tfrac{1}{2} = 12; \\ y &= 6 \div \tfrac{1}{4} = 24; \\ z &= 6 \div \tfrac{1}{11} = 66. \end{aligned}\right\} \text{ Ans.}$$

4. Let x = 1st part;
y = 2d "
z = 3d "
u = 4th "
v = 5th "
then $x : 1 = y : 2 = z : 3 = u : 4 = v : 5;$
or $x : y : z : u : v = 1 : 2 : 3 : 4 : 5.$
By Theorem IV.,

$$\frac{x}{1} : \frac{y}{2} : \frac{z}{3} : \frac{u}{4} = \frac{v}{5} = \frac{x+y+z+u+v}{1+2+3+4+5} = \frac{1000}{15} = \frac{200}{3}.$$

$$\left.\begin{aligned} \text{Hence } x &= \tfrac{200}{3} \div 1 = \tfrac{200}{3} = 66\tfrac{2}{3}; \\ y &= \tfrac{200}{3} \div \tfrac{1}{2} = \tfrac{400}{3} = 133\tfrac{1}{3}; \\ z &= \tfrac{200}{3} \div \tfrac{1}{3} = \tfrac{600}{3} = 200; \\ u &= \tfrac{200}{3} \div \tfrac{1}{4} = \tfrac{800}{3} = 266\tfrac{2}{3}; \\ v &= \tfrac{200}{3} \div \tfrac{1}{5} = \tfrac{1000}{3} = 333\tfrac{1}{3}. \end{aligned}\right\} \text{ Ans.}$$

5. Let x = 1st fraction;
y = 2d "
then $x : y = a : b.$
By Theorem IV.,

$$\frac{x}{a} = \frac{y}{b} = \frac{x+y}{a+b} = \frac{1}{a+b}.$$

$$\left.\begin{aligned} \text{Hence } x &= \frac{1}{a+b} \div \frac{1}{a} = \frac{a}{a+b}; \\ y &= \frac{1}{a+b} \div \frac{1}{b} = \frac{b}{a+b}. \end{aligned}\right\} \text{ Ans.}$$

6. Let x = 1st number;
y = 2d "
then $x : y = 7 : 3.$

By Theorem IV.,

$$\frac{x}{7} = \frac{y}{3} = \frac{x-y}{7-3} = \frac{24}{4} = 6.$$

Hence $\left.\begin{array}{l} x = 6 \div \frac{1}{7} = 42; \\ y = 6 \div \frac{1}{3} = 18. \end{array}\right\}$ Ans.

7. Let $x =$ 1st number;
$y =$ 2d "
then $x : y = m : n.$
By Theorem IV.,

$$\frac{x}{m} = \frac{y}{n} = \frac{x-y}{m-n} = \frac{1}{m-n}.$$

$$\left.\begin{array}{l} x = \dfrac{1}{m-n} \div \dfrac{1}{m} = \dfrac{m}{m-n}; \\ y = \dfrac{1}{m-n} \div \dfrac{1}{n} = \dfrac{n}{m-n}. \end{array}\right\} \text{Ans.}$$

8. Multiplying extremes and means,

$$bx = ay;$$

transpose, $bx - ay = 0$ (1);
$\times$ (1) by $a = abx - a^2y = 0$ (2);
$\times$ given equation by b,

$$abx - b^2y = ab + b^2 \text{ (3)};$$

$(3) - (2) = a^2y - b^2y = ab + b^2;$

$$(a^2 - b^2)\, y = b\,(a + b);$$

$$y = \frac{b\,(a+b)}{a^2 - b^2} = \frac{b}{a-b}.$$

Sub. in (1), $bx - \dfrac{ab}{a-b} = 0;$

transpose, $bx = \dfrac{ab}{a-b};$

$$x = \frac{ab}{b\,(a-b)} = \frac{a}{a-b}.$$

9. By Theorem I.,

$$aB = Ab;$$

and $cD = Cd.$

Multiplying member by member,

$$aBcD = AbCd,$$

or $ac \times BD = bd \times AC;$

$\div DB$, $ac = \dfrac{bd \times AC}{BD};$

$$\div bd, \ \frac{ac}{bd} = \frac{AC}{BD};$$
$$\text{or } ac : bd :: AC : BD.$$

10. Adding and subtracting $2y$,
$$x + 2y = ay + 2y = (a+2)y;$$
$$x - y = ay - 2y = (a-2)y;$$
$$\therefore \ \frac{x+2y}{x-2y} = \frac{a+2}{a-2}.$$

11. Cl. of fractions, $x + 2y = 5x - 10y$;
$$\therefore \ 4x = 12y \text{ and } x = 3y.$$
Adding and subtracting y,
$$x + y = 4y; \ x - y = 2y;$$
$$\therefore \ \frac{x+y}{x-y} = 2. \quad \text{Ans.}$$

12. By Theorem V., $a^2 : b^2 = p^2 : q^2$.
By Theorem III., $a^2 + b^2 : a^2 = p^2 + q^2 : p^2$ (1);
also, by Theorem III., $a + b : a = p + q : p$;
$$\text{or } a : a + b = p : p + q;$$
$$\text{or } \frac{a}{a+b} = \frac{p}{p+q} \quad (2).$$
Multiplying the consequents of (1) by (2),
$$a^2 + b^2 : \frac{a^2}{a+b} = p^2 + q^2 : \frac{p^2}{p+q}.$$
Also, from Theorems V. and III.,
$$a^n + b^n : a^n = p^n + q^n : p^n.$$
Multiplying the consequents by the members of (2) we have the result to be proven.

13. By Theorem III.,
$$(a+b+c+d) + (a+b-c-d) : (a+b+c+d) - (a+b-c-d)$$
$$= (a-b+c-d) + (a-b-c+d) : (a-b+c-d) - (a-b-c+d);$$
$$\text{or } 2a + 2b : 2c + 2d = 2a - 2b : 2c - 2d;$$
$$\div 2 = a + b : c + d = a - b : c - d.$$
By Theorem II., $a + b : a - b = c + d : c - d$.
By Theorem III., $(a+b) + (a-b) : (a+b) - (a-b)$
$$= c + d + (c-d) : (c+d) - (c-d);$$
$$\text{or } 2a : 2b = 2c : 2d;$$
$$\div 2 = a : b = c : d.$$

14.

$$\begin{aligned}
\text{Let } x &= \text{one share;}\\
2x &= \text{A's} \quad \text{"}\\
3x &= \text{B's} \quad \text{"}\\
7x &= \text{C's} \quad \text{"}
\end{aligned}$$

then $7x - 1256 = 3x + 1256$;

trans., $7x - 3x = 1256 + 1256$;

$$\begin{aligned}
4x &= 2512;\\
x &= 628;\\
2x &= 1256;\\
3x &= 1884;\\
7x &= 4396. \quad \text{Total, \$7536. Ans.}
\end{aligned}$$

15.

$$\begin{aligned}
\text{Let } x &= \text{one share of profit, 1st year;}\\
2x &= \text{A's;}\\
5x &= \text{B's.}\\
\text{Let } y &= \text{one share, 2d year;}\\
3y &= \text{A's;}\\
4y &= \text{B's.}
\end{aligned}$$

Then $3y = 2x + 3200$;

$4y = 5x + 1700$.

Transpose, $3y - 2x = 3200$ (1);

$4y - 5x = 1700$ (2);

$\times$ (1) by 4 $= 12y - 8x = 12\,800$ (3);

$\times$ (2) by 3 $= 12y - 15x = 5\,100$ (4);

(4) $-$ (3) $= 7x = 7700 = 2x + 5x =$ profit 1st year.

$\times$ (1) by 5 $= 15y - 10x = 16\,000$ (5);

$\times$ (2) by 2 $= 8y - 10x = 3\,400$ (6);

(5) $-$ (6) $= 7y = 12\,600 = 3y + 4y =$ profit 2d yr.

16. The ratio of chickens to ducks is $\frac{7}{2}$, or $\frac{c}{d} = \frac{7}{2}$;

also, $\frac{d}{g} = \frac{3}{2}$;

$\frac{c}{d} \times \frac{d}{g} = \frac{c}{g} = \frac{21}{4}$.

That is, there were 4 geese to every 21 chickens.

Hence $\times$ 2, there were 8 geese to 42 chickens.

17.

$h : c = 4 : 9$;

$h - 148 : c - 108 = 1 : 3$.

From first proportion,

$9h = 4c$, or $9h - 4c = 0$ (1).

From second, $3h - 444 = c - 108$, or $3h - c = 336$ (2);

$\times$ (2) by 3 $= 9h - 3c = 1008$ (3);

(3) $-$ (1) $= c = 1008$.

Substitute, $9h = 4c = 4032$;

$h = 448$.

18. If there are a parts water and b parts wine,

$$a + b \text{ will be the whole.}$$

$\dfrac{a}{a+b} =$ the ratio of water to the whole;

$\dfrac{b}{a+b} =$ " " " wine " " "

$\dfrac{a}{a+b} + \dfrac{b}{a+b} =$ the sum $= 1$.

19. Let x be the equal weight taken from each ingot. Then from the first ingot I take

$\frac{1}{2} x$ of gold, $\frac{1}{2} x$ of silver.

From the second, $\frac{2}{3} x$ of gold, $\frac{1}{3} x$ of silver.

The combination therefore contains $\frac{7}{6} x$ of gold, $\frac{5}{6} x$ of silver.

The ratio is $7 : 5$; that is, 7 parts of gold to 5 of silver.

20. Since, in the first ingots, the quantities are equal, 1 ounce from it will contain

$$\tfrac{1}{2} \text{ oz. gold, } \tfrac{1}{2} \text{ oz. silver.}$$

Since the second has two parts of gold to one of silver, the alloy from it will be $\frac{2}{3}$ gold and $\frac{1}{3}$ silver. This alloy, being 3 ounces, will have

$$2 \text{ oz. gold, } 1 \text{ oz. silver.}$$

Therefore both alloys together will contain

$\frac{1}{2} + 2 = \frac{5}{2}$ oz. gold and $\frac{1}{2} + 1 = \frac{3}{2}$ oz. silver.

The ratio is

$$\text{gold} : \text{silver} = 5 : 3.$$

21. As in Problem 18, one gallon from first cask will contain

$\dfrac{a}{a+b}$ gallon of water and $\dfrac{b}{a+b}$ gallon of alcohol.

One gallon from the second will contain

$\dfrac{m}{m+n}$ gallon of water and $\dfrac{n}{m+n}$ gallon of alcohol.

The mixture will contain

$\dfrac{a}{a+b} + \dfrac{m}{m+n} = \dfrac{2am + an + bm}{(a+b)(m+n)}$ of a gallon of water

and

$\dfrac{b}{a+b} + \dfrac{n}{m+n} = \dfrac{2bn + an + bm}{(a+b)(m+n)}$ of a gall. of alcohol.

22. Two parts from the first cask will contain

$\dfrac{2a}{a+b}$ parts of water and $\dfrac{2b}{a+b}$ parts of alcohol.

One equal part from the second will contain
$\frac{m}{m+n}$ of one part water and $\frac{n}{m+n}$ of one part alcohol.
The mixture will contain

$$\frac{2a}{a+b}+\frac{m}{m+n}=\frac{3am+2an+bm}{(a+b)(m+n)} \text{ parts of water}$$

and

$$\frac{2b}{a+b}+\frac{n}{m+n}=\frac{3bn+an+2bm}{(a+b)(m+n)} \text{ parts of alcohol.}$$

Dividing, we find the ratio to be
water : alcohol $= 3am+2an+bm : 3bn+an+2bm.$

23. Reasoning as in the preceding problem, we find:
From 1st cask $\frac{pa}{a+b}$ parts water and $\frac{pb}{a+b}$ pts. alcohol;
" 2d " $\frac{qm}{m+n}$ " " " $\frac{qn}{m+n}$ " "

The whole, $\frac{(p+q)am+pan+qbm}{(a+b)(m+n)}$ parts of water
and $\frac{(p+q)bn+pbm+qan}{(a+b)(m+n)}$ parts of alcohol.
Whence, for the ratio, water : alcohol
$=(p+q)am+pan+qbm : (p+q)bn+pbm+qan.$

24. If r be the fraction of gold in the first ingot, and s in the second ingot, then $1-r$ will be the fraction of silver in the first, and $1-s$ in the second. Therefore 2 parts from first ingot contain
$2r$ parts of gold and $2(1-r)$ parts of silver.
1 part from the second contains
s parts of gold and $(1-s)$ parts of silver.
The combination will therefore contain
$(2r+s)$ p. of gold and $2(1-r)+(1-s)$ p. of silver.
Since these quantities are equal, we have the equation

$$\begin{aligned} 2r+s &= 2(1-r)+(1-s) \\ &= 3-2r-s, \end{aligned}$$

or $$4r+2s=3. \qquad (a)$$

Proceeding in the same way with the second mixture, we find that one part of the first ingot will contain
r of gold and $1-r$ of silver;
two parts from the second will contain
$2s$ of gold and $2(1-s)$ of silver.
So the second combination will contain
$(r+2s)$ parts of gold and $1-r+2(1-s)$ of silver.

Equating the ratio of these quantities to $3:5$, we have

$$5r + 10s = 3 - 3r + 6 - 6s,$$

or $\qquad 8r + 16s = 9 \qquad (b)$

$(a) \times 2 = \qquad 8r + 4s = 6$

$\qquad 12s = 3 \qquad \therefore \quad s = \frac{1}{4}.$

Sub. in (a), $\qquad 4r + \frac{1}{2} = 3;$

$\qquad 4r = \frac{5}{2};$

$\qquad r = \frac{5}{8}.$

Therefore the first ingot is $\frac{5}{8}$ gold and $\frac{3}{8}$ silver; the second is $\frac{1}{4}$ gold and $\frac{3}{4}$ silver.

25. The mistake would consist in not having the quantities of gold, $2p$ and r, taken from the two ingots expressed in the same unit. If we suppose the word "part" to express the same quantity throughout, and say, as in the example,

$$2 \text{ parts} = 2p + 2q, \qquad (1)$$

the units in which the second member is expressed must be different from a "part" in the first member, unless $2p + 2q = 2$, or $p + q = 1$. In the same way, when we say

$$1 \text{ part} = r + s, \qquad (2)$$

the unit of the second member will not be a part unless $r + s = 1$.

If we suppose $p + q$ and $r + s$ different from unity, the unit of second member of (1) will be

$$\frac{1 \text{ part}}{p+q},$$

and that of (2) will be

$$\frac{1 \text{ part}}{r+s};$$

and if $p + q = r + s$, these units will be equal and the quantities may be added.

26. Let $a : b :: p : q$,
and $p : b :: b : q$, or $b^2 = pq$ (1).
By Theorem I., $\qquad ad = bc;$

$$\div a,\ d = \frac{bc}{a};$$

$$\times c,\ cd = \frac{bc^2}{a}.$$

Since the question states a certain relation between the first three terms of the proposition, we must eliminate the last term q. From (1) we have

$$q = \frac{b^2}{p}.$$

Substituting this in the original proportion, we have

$$a : b = p : \frac{b^2}{p}.$$

Equating products of extremes and means,

$$\frac{ab^2}{p} = bp;$$

whence

$$ab = p^2,$$

and $p = \sqrt{ab}$, the mean proportional between a and b as asserted.

Sub. this value of cd in (1),

$$b^2 = \frac{bc^2}{a};$$

$$\div\, b,\ b = \frac{c^2}{a},\ \text{or } c^2 = ab. \quad \therefore (2) \text{ is true.}$$

§ 167.

1. $a^2b^2c^2,\ a^n b^n c^n.$
2. $\frac{a^2b^2}{c^2},\ \frac{a^n b^n}{c^n}.$
3. $a^2b^2c^{-2},\ a^n b^n c^{-n}.$
4. $\frac{m^2n^2}{p^2q^2},\ \frac{m^n n^n}{p^n q^n}.$
5. $\frac{(a+b)^2}{(a-b)^2},\ \frac{(a+b)^n}{(a-b)^n}.$
6. $\frac{m^2n^2\,(a+b)^2}{p^2q^2\,(a-b)^2},\ \frac{m^n n^n\,(a+b)^n}{p^n q^n\,(a-b)^n}.$

§ 168.

1. $27\,x^3y^6.$
2. $\frac{64\,a^3}{b^3}.$
3. $a^{3m}.$
4. $b^3x^{12}.$
5. $8\,a^6m^{3n}.$
6. $\frac{216\,a^{3m}}{b^3}.$
7. $a^n.$
8. $a^{2n}b^n.$
9. $a^{2n}b^{2n}c^n.$
10. $a^{mn}\,a^{n^2} = a^{m+n^2}.$
11. $2^n p^{mn} q^{2n}.$
12. $(a+b)^n\,(c+d)^n.$
13. $(x+y)^n\,(x-y)^n.$
15. $\frac{a^n}{b^n}.$
16. $\frac{a^{2n}}{b^{2n}}.$

17. $\dfrac{(x+y)^n}{(x-y)^n}.$

19. $\dfrac{a^n b^n (c-d)^{2n}}{(a-b)^n c^{3n}}.$

20. $8a^3b^6m^9.$

21. $9m^2n^2x^4.$

22. $2a(-27b^6m^3n^9)$ or $-54ab^6m^3n^9.$

23. $2401p^4q^8r^{12}.$

24. $a^i b^{ni}.$

25. $2^n a^{2n}x^{3n}.$

26. $m^{n^2}.$

§ 169.

1. $a^6b^{-6} = \dfrac{a^6}{b^6}.$

2. $a^{12}b^{-12} = \dfrac{a^{12}}{b^{12}}.$

3. $a^6m^6p^{-18} = \dfrac{a^6m^6}{p^{18}}.$

4. $a^{-6m}b^{-6n} = \dfrac{1}{a^{6m}b^{6n}}.$

5. $(a+b)^{18}(a-b)^{-18} = \dfrac{(a+b)^{18}}{(a-b)^{18}}.$

6. $(x+y)^{6n}(x+z)^{-6n} = \dfrac{(x+y)^{6n}}{(x+z)^{6n}}.$

7. $\dfrac{a^{-6p}}{b^{-6q}} = \dfrac{b^{6q}}{a^{6p}}.$

8. $\dfrac{(a+b)^{-6m}}{(a-b)^{-6n}} = \dfrac{(a-b)^{6n}}{(a+b)^{6m}}.$

9. $(a+b)^{-n}(a-b)^n = \dfrac{(a-b)^n}{(a+b)^n}.$

10. $a^5b^{-5}c^{-10} = \dfrac{a^5}{b^5c^{10}}.$

11. $a^{-5}b^5c^{10} = \dfrac{b^5c^{10}}{a^5}.$

12. $m^{-i^2}n^{ij} = \dfrac{n^{ij}}{m^{i^2}}.$

13. $x^{-i^2}y^{i^2} = \dfrac{y^{i^2}}{x^{i^2}}.$

14. $a^{0n}b^{n^2}c^{-n^2} = \dfrac{a^{0n}b^{n^2}}{c^{n^2}} = \dfrac{b^{n^2}}{c^{n^2}}.$

§ 170.

1. 4.

2. $-27.$

3. 256.

4. 25.

5. $-125.$

6. $-b^7.$

7. $-(a+b)^3$.

8. $-m^7n^7$.

9. p^6q^6.

10. a^{2n}.

11. $-b^{2n+1}$.

12. $-(a+b)^{2n-1}$.

13. $1^{2n}=1$.

14. $-1^{2n+1}=-1$.

15. $-1^{2n-1}=-1$.

§ 172.

3. The exponents of a are 9, 8, 7, 6, 5, 4, 3, 2, 1, 0;
" " " b " 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
Coefficient of 1st term is 1;
" " 2d " " $n=9$;
" " 3d " " $\frac{9-1}{2}.9=36$;
" " 4th " " $\frac{9-2}{3}.36=84$;
" " 5th " " $\frac{9-3}{4}.84=126$;
" " 6th " " $\frac{9-4}{5}.126=126$.
" " 7th " " $\frac{9-5}{6}.126=84$;
" " 8th " " $\frac{9-6}{7}.84=36$;
" " 9th " " $\frac{9-7}{8}.36=9$;
" " 10th " " $\frac{9-8}{9}.9=1$.

$$\therefore (a+b)^9 = a^9+9a^8b+36a^7b^2+84a^6b^3+126a^5b^4 + 126a^4b^5+84a^3b^6+36a^2b^7+9ab^8+b^9.$$

4. b^3 is in the 4th term, and the coefficient is
$$\frac{n(n-1)(n-2)}{2.3}=\frac{10.9.8}{2.3}=120. \quad \text{Ans.}$$

5. $=(2am)^8+8(2am)^7(3n)+28(2am)^6(3n)^2 + 56(2am)^5(3n)^3$;
$=256a^8m^8+3072a^7m^7n+16\,128a^6m^6n^2 + 48\,384a^5m^5n^3$.

6. $1^{18}+18.1^{17}\left(\frac{x}{y}\right)+153.1^{16}\left(\frac{x}{y}\right)=1+18\frac{x}{y}+153\frac{x^2}{y^2}$.

7. $a^{13} - 13a^{12}x + 78a^{11}x^2$. The coefficients of the last terms are the same as the coefficients of the corresponding first terms; the exponents only change places, and the sign of the last term is negative. We get

$$-78a^2x^{11} + 13ax^{12} - x^{13}.$$

8. $$a^6 + 6.a^5\left(\frac{1}{a}\right) + 15.a^4\left(\frac{1}{a}\right)^2 + 20a^3\left(\frac{1}{a}\right)^3 + 15a^2\left(\frac{1}{a}\right)^4 + 6a\left(\frac{1}{a}\right)^5 + \left(\frac{1}{a}\right)^6 = a^6 + 6a^4 + 15a^2 + 20 + \frac{15}{a^2} + \frac{6}{a^4} + \frac{1}{a^6}.$$

9. I. $$= 1 + n.x^2 + \frac{n(n-1)}{2}x^4 + \frac{n(n-1)(n-2)}{2.3}x^6.$$

II. $$= 1^n + n.1^{n-1}(2x^2) + \frac{n(n-1)}{2}1^{n-2}(2x^2)^2 + \frac{n(n-1)(n-2)}{2.3}1^{n-3}(2x^2)^3.$$

$$= 1 + 2n.x^2 + 2n(n-1)x^4 + \frac{4n(n-1)(n-2)}{3}x^6.$$

III. $$= 1 - n.1^{n-1}(2x^2) + \frac{n(n-1)}{2}1^{n-2}(2x^2)^2 - \frac{n(n-1)(n-2)}{2.3}1^{n-3}(2x^2)^3.$$

$$= 1 - 2nx^2 + 2n(n-1)x^4 - \frac{4n(n-1)(n-2)}{3}x^6.$$

IV. $$= \left(\frac{1}{ax}\right)^8 + 8\left(\frac{1}{ax}\right)^7 a + 28\left(\frac{1}{ax}\right)^6 a^2 + 56\left(\frac{1}{ax}\right)^5 a^3 = \frac{1}{a^8x^8} + \frac{8}{a^6x^7} + \frac{28}{a^4x^6} + \frac{56}{a^2x^5}.$$

V. $$= \left(7\frac{y^2}{x^2}\right)^5 - 5\left(7\frac{y^2}{x^2}\right)^4\left(8\frac{x^2}{y^2}\right) + 10\left(7\frac{y^2}{x^2}\right)^3\left(8\frac{x^2}{y^2}\right)^2 - 10\left(7\frac{y^2}{x^2}\right)^2\left(8\frac{x^2}{y^2}\right)^3 = 16\,807\frac{y^{10}}{x^{10}} - 96\,040\frac{y^6}{x^6} + 219\,520\frac{y^2}{x^2} - 250\,880\frac{x^2}{y^2}.$$

VI. $$= (3am^{\frac{1}{2}})^{10} - 10(3am^{\frac{1}{2}})^9(5bn^{\frac{1}{2}}) + 45(3am^{\frac{1}{2}})^8(5bn^{\frac{1}{2}})^2 - 120(3am^{\frac{1}{2}})^7(5bn^{\frac{1}{2}})^3 = 59\,049a^{10}m^5 - 984\,150a^9m^{\frac{9}{2}}bn^{\frac{1}{2}} + 7\,381\,125a^8m^4b^2n - 32\,805\,000a^7m^{\frac{7}{2}}b^3n^{\frac{3}{2}}.$$

10. $= 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$
$\quad 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$
Sum $= 2 + 42x^2 + 70x^4 + 14x^6$.
Difference $= 14x + 70x^3 + 42x^5 + 2x^7$.

§ 173.

1. $= 1 + 4x^2 + 9x^4 + 4x + 6x^2 + 12x^3$
$= 1 + 4x + 10x^2 + 12x^3 + 9x^4$.

2. $= 1 + 4x^2 + 9x^4 + 16x^6 + 4x + 6x^2 + 12x^3 + 8x^3 + 16x^4 + 24x^5 = 1 + 4x + 10x^2 + 20x^3 + 25x^4 + 24x^5 + 16x^6$.

3. $= 1 + 4x^2 + 9x^4 + 16x^6 + 25x^{10} + 4x + 6x^2 + 12x^3 + 8x^3 + 16x^4 + 24x^5 + 10x^5 + 20x^6 + 30x^7 + 40x^8$
$= 1 + 4x + 10x^2 + 20x^3 + 25x^4 + 34x^5 + 36x^6 + 30x^7 + 40x^8 + 25x^{10}$.

4. $= 1 + 4x^2 + 9x^4 + 16x^6 + 25x^{10} + 36x^{12} + 4x + 6x^2 + 12x^3 + 8x^3 + 16x^4 + 24x^5 + 10x^5 + 20x^6 + 30x^7 + 40x^8 + 12x^6 + 24x^7 + 36x^8 + 48x^9 + 60x^{11}$
$= 1 + 4x + 10x^2 + 20x^3 + 25x^4 + 34x^5 + 48x^6 + 54x^7 + 76x^8 + 48x^9 + 25x^{10} + 60x^{11} + 36x^{12}$.

5. $= 1 + 4x^2 + 9x^4 + 16x^6 - 4x + 6x^2 - 12x^3 - 8x^3 + 16x^4 - 24x^5$
$= 1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6$.

6. $= a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2bc - 2ad + 2bd - 2cd$.

7. $= 9a^2 + 4b^2 + c^2 + d^2 + 2(3a)(2b) - 2c(3a) - 2c(2b) + 2d(3a) + 2d(2b) - 2cd = 9a^2 + 4b^2 + c^2 + d^2 + 12ab - 6ac - 4bc + 6ad + 4bd - 2cd$.

8. $= a^2 + \dfrac{1}{a^2} + b^2 + \dfrac{1}{b^2} + 2a\left(\dfrac{1}{a}\right) - 2ab - 2b\left(\dfrac{1}{a}\right) - 2a\left(\dfrac{1}{b}\right) - 2\left(\dfrac{1}{a}\right)\left(\dfrac{1}{b}\right) + 2b\left(\dfrac{1}{b}\right)$

$= a^2 + \dfrac{1}{a^2} + b^2 + \dfrac{1}{b^2} + 2 - 2ab - \dfrac{2b}{a} - \dfrac{2a}{b} - \dfrac{2}{ab} + 2$

$= (a - b)^2 + 4 + \dfrac{(a - b)^2 - 2ab(a^2 - b^2)}{a^2b^2}$.

§ 176.

1. $m^{\frac{1}{2}}$.

2. $(m + n)^{\frac{1}{2}}$.

3. $(a + b)^{\frac{1}{3}}$.

4. $(a + b)^{\frac{2}{3}}$.

5. $m^{\frac{3}{2}}$.

6. $a^{\frac{n}{5}}$.

7. $a^{\frac{5}{n}}$.

8. $(a + b)^{\frac{n}{m}}$.

9. $(a + b)^{\frac{m}{n}}$.

§ 177.

1. $= \pm (a+b)^{\frac{3}{2}}, \qquad a+b, \qquad (a+b)^{\frac{3}{n}}.$

2. $= \pm (a+b), \qquad (a+b)^{\frac{2}{3}}, \qquad (a+b)^{\frac{2}{n}}.$

3. $= \pm (a+b)^{\frac{1}{2}}, \qquad (a+b)^{\frac{1}{3}}, \qquad (a+b)^{\frac{1}{n}}.$

4. $= \pm (x+y)^{\frac{3}{4}}, \qquad (x+y)^{\frac{1}{2}}, \qquad (x+y)^{\frac{3}{2n}}.$

5. $= \pm (x+y)^{\frac{1}{4}}, \qquad (x+y)^{\frac{1}{6}}, \qquad (x+y)^{\frac{1}{2n}}.$

6. $= \pm (x+y)^{\frac{1}{2n}}, \qquad (x+y)^{\frac{1}{3n}}, \qquad (x+y)^{\frac{1}{n^2}}.$

§ 178.

1. $= 2x.$

2. $= \dfrac{3ax}{7m^{\frac{1}{2}}}.$

3. $= \dfrac{8a^{\frac{1}{3}}bc^{\frac{2}{3}}}{9m^{\frac{1}{3}}pq^{\frac{1}{3}}}.$

4. $= 3.4 = 12.$

5. $= 3a.$

6. $= 4.3.ab^2 = 12ab^2.$

7. $= a^{\frac{1}{2}}b^{\frac{3}{2}}cd^{\frac{1}{2}}.$

8. $= \dfrac{2a^{\frac{m}{3}}}{5x^{\frac{1}{3}}y^{\frac{n}{3}}}.$

9. $= 7^{\frac{1}{n}}.$

10. $= 4^{\frac{1}{n}}.7^{\frac{1}{n}} = 28^{\frac{1}{n}}.$

11. $= 4^{\frac{1}{n}}.7^{\frac{1}{n}}.10^{\frac{1}{n}} = 280^{\frac{1}{n}}.$

12. $= \dfrac{5^{\frac{1}{n}}a^{\frac{1}{n}}b}{6^{\frac{1}{n}}m^{\frac{1}{n}}p}.$

13. $= 6^{\frac{1}{n}}ab^3.$

14. $= \dfrac{6^{\frac{1}{n}}a^{\frac{2}{n}}b^{\frac{1}{2}}}{c^{\frac{m}{n}}d^{\frac{m}{n^2}}}.$

15. $= \dfrac{x^{\frac{m+1}{n}}yz^{\frac{m-2}{n}}}{a^m b^{\frac{1}{m}}}.$

16. $= 3^6a^{-2}(a+b)^4(x-y).4.(b-c+d)^{-4}.$

17. $= a^{\frac{1}{n}}(b-c)^{\frac{m}{n}}.$

18. $= a^{\frac{1}{m}}b^{\frac{2}{m}}c^{\frac{3}{m}}.$

19. $= ab^{\frac{q}{p}}.$

20. $= \dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$

21. $= \dfrac{(a+b)^{\frac{n}{m}}}{(a-b)^{\frac{n}{m}}}.$

§ 179.

1. $= a,\qquad a^{\frac{3}{2}},\qquad a^{\frac{n}{2}}.$

2. $= a^{\frac{2}{3}},\qquad a,\qquad a^{\frac{n}{3}}.$

3. $= a^{\frac{4}{3}},\qquad a^{2},\qquad a^{\frac{2n}{3}}.$

4. $= a^{\frac{2}{n}},\qquad a^{\frac{3}{n}},\qquad a.$

5. $= a^{2}b^{\frac{2}{n}},\qquad a^{3}b^{\frac{3}{n}},\qquad a^{n}b.$

6. $= a^{2}b^{\frac{2}{3}}c^{n},\qquad a^{3}bc^{\frac{3n}{2}}\qquad a^{n}b^{\frac{n}{3}}c^{\frac{n^2}{2}}.$

7. $= a^{m}b^{\frac{2m}{3}},\qquad a^{\frac{3m}{2}}b^{m},\qquad a^{\frac{mn}{2}}b^{\frac{mn}{3}}.$

8. $= a^{\frac{2p}{q}}b^{-\frac{2q}{p}},\qquad a^{\frac{3p}{q}}b^{-\frac{3q}{p}},\qquad a^{\frac{np}{q}}b^{-\frac{nq}{p}}.$

9. $= (a+b)^{\frac{2m}{n}}(a-b)^{-2n},\quad (a+b)^{\frac{3m}{n}}(a-b)^{-3n},$
$(a+b)^{m}(a-b)^{-n^2}.$

10. $= a^{-2n}b^{2n},\qquad a^{-3n}b^{3n},\qquad a^{-n^2}b^{n^2}.$

11. $= a^{-\frac{2}{n}}b^{\frac{2}{n}}\qquad a^{-\frac{3}{n}}b^{\frac{3}{n}},\qquad a^{-1}b.$

12. $= \dfrac{(x+y)^{-\frac{2}{p}}}{(x-y)^{-\frac{2m}{q}}},\qquad \dfrac{(x+y)^{-\frac{3}{p}}}{(x-y)^{-\frac{3m}{q}}},\qquad \dfrac{(x+y)^{-\frac{n}{p}}}{(x-y)^{-\frac{mn}{q}}}.$

13. $= x^{\frac{pm}{n}}y^{-\frac{\quad}{n}}.$

14. $= a^{\frac{2m}{3n}}b^{\frac{4m}{3n}}c^{-\frac{6m}{5n}}.$

15. $= a^{-\frac{q}{2}}b^{-\frac{3q}{4}}.$

16. $= a^{\frac{pm}{qn}}.$

17. $= \dfrac{x^{-\frac{2}{3}}}{y^{-1}}.$

18. $= \dfrac{a^{2m}}{b^{2m-\frac{1}{2}}}.$

§ 181.

1. $= 13\sqrt{3} + (7ab - 5)\sqrt{2}.$

2. $= [6 + 2(a+b)]\sqrt{x+y} + [a - 3(a+b)]\sqrt{x-y}.$

3. $= a\sqrt{7}.$ 4. $= 2a\sqrt{xy}.$

5. $= (a - b + b - c + c - a)\sqrt{x} = 0.$

6. $= (a - 1 + 2a - a - b)\sqrt{x} = (2a - 1 - b)\sqrt{x}.$

7. $= (\frac{3}{4} - a + 6 - c + \frac{1}{3})\sqrt{x} = (7\frac{1}{12} - a - c)\sqrt{x}.$

8. $= \left(\frac{a+b}{2} - 6c - \frac{a+b}{3} + 1\right)\sqrt{x} = \left(\frac{a+b}{6} - 6c + 1\right)\sqrt{x}.$

9. $= \left(\frac{3}{4} - 1 + a - b + \frac{2(a-b)}{3}\right)\sqrt{x}$

$= \left(\frac{3a - 3b}{3} + \frac{2a - 2b}{3} - \frac{1}{4}\right)\sqrt{x} = \left(\frac{5a - 5b}{3} - \frac{1}{4}\right)\sqrt{x}.$

10. $= \left(1 - b + \frac{6(a-b)}{4} - \frac{1}{2}\right)\sqrt{a} - \sqrt{x}$

$= \left(\frac{1}{2} - \frac{4b}{4} + \frac{6a - 6b}{4}\right)\sqrt{a} - \sqrt{x}$

$= \left(\frac{1}{2} + \frac{6a - 10b}{4}\right)\sqrt{a} - \sqrt{x} = \frac{1 + 3a - 5b}{2}\sqrt{a} - \sqrt{x}.$

11. $= \left(\frac{3}{4} - 1 + \frac{2(a-b)}{3}\right)\sqrt{x} = \left(\frac{2(a-b)}{3} - \frac{1}{4}\right)\sqrt{x}.$

12. $= (4 - \frac{1}{3} + a - b)\sqrt{x} = (3\frac{2}{3} + a - b)\sqrt{x}.$

§ 182.

2. $= \sqrt{7.5} = \sqrt{35}.$ 3. $= \sqrt{7a}.$

4. $= \sqrt{a(a+y)} = \sqrt{a^2 + ay}.$

5. $= \sqrt{ab(a+b)} = \sqrt{a^2b + ab^2}.$

6. $= \sqrt{(x+1)(x-1)} = \sqrt{x^2 - 1}.$

7. $= \sqrt{(x^2+1)(x+1)(x-1)} = \sqrt{x^4-1}.$

8. $= [(a^2-b^2)^{\frac{1}{4}}]^2 = (a^2-b^2)^{\frac{1}{2}}.$

9. $= [(x^2+1)(x+1)(x-1)]^{\frac{1}{3}} = (x^4-1)^{\frac{1}{3}}.$

§ 183.

1. $= \sqrt{4\times 2} = 2\sqrt{2}.$ **4.** $= \sqrt{81} = 9.$

2. $= \sqrt{16\times 2} = 4\sqrt{2}.$ **5.** $= \sqrt{a^2b^2c^2} = abc.$

3. $= \sqrt{64\times 2} = 8\sqrt{2}.$ **6.** $= \sqrt{144} = 12.$

7. $= \sqrt{288} = \sqrt{144\times 2} = 12\sqrt{2}.$

8. $= \sqrt{x^2+2x+1} = x+1.$

9. $= \sqrt{25\times 7} = 5\sqrt{7}.$ **11.** $= \sqrt{36\times 3} = 6\sqrt{3}.$

10. $= \sqrt{25\times 6} = 5\sqrt{6}.$ **12.** $= x\sqrt{a+b}.$

13. $= \sqrt{x(a^2+2ab+b^2)} = (a+b)\sqrt{x}.$

14. $= \sqrt{y(a^2+4a+4)} = (a+2)\sqrt{y}.$

15. $= \sqrt{z(4m^2+8m+4)} = (2m+2)\sqrt{z}.$

16.
$$\begin{array}{rlrlr} 4\sqrt{2} & = & & & 4\sqrt{2} \\ -6\sqrt{8} & = & -6\sqrt{4\times 2} & = -6.2\sqrt{2} = & -12\sqrt{2} \\ 10\sqrt{32} & = & 10\sqrt{16\times 2} & = 10.4\sqrt{2} = & 40\sqrt{2} \\ \hline & & & & 32\sqrt{2}.\ \text{Ans.} \end{array}$$

17.
$$\begin{array}{rlrl} \sqrt{12} & = \sqrt{4\times 3} & = & 2\sqrt{3} \\ \sqrt{27} & = \sqrt{9\times 3} & = & 3\sqrt{3} \\ \sqrt{75} & = \sqrt{25\times 3} & = & 5\sqrt{3} \\ \hline & & & 10\sqrt{3}.\quad \text{Ans.} \end{array}$$

18. $\sqrt{4a} = \sqrt{4\times a} = 2\sqrt{a}$

$2\sqrt{a} - 2\sqrt{a} = 0.$

19.
$$\begin{array}{rlrl} \sqrt{125} & = \sqrt{25\times 5} & = & 5\sqrt{5} \\ -\sqrt{45} & = -\sqrt{9\times 5} & = & -3\sqrt{5} \\ -\sqrt{80} & = -\sqrt{16\times 5} & = & -4\sqrt{5} \\ \hline & & & -2\sqrt{5}.\quad \text{Ans.} \end{array}$$

20.
$$\begin{aligned} \sqrt[3]{81} &= -\sqrt[3]{27 \times 3} = 3\sqrt[3]{3} \\ -\sqrt[3]{192} &= -\sqrt[3]{64 \times 3} = \underline{-4\sqrt[3]{3}} \\ & \qquad\qquad -\sqrt[3]{3}. \quad \text{Ans.} \end{aligned}$$

21.
$$\begin{aligned} \sqrt[3]{a^2b^3} &= b\sqrt[3]{a^2} \\ \sqrt[3]{a^2c^6} &= c^2\sqrt[3]{a^2} \\ & (b + c^2)\sqrt[3]{a^2}. \quad \text{Ans.} \end{aligned}$$

§ 184.

1.
$$\begin{aligned} &= 2(5 - 3\sqrt{2}) = 10 - 6\sqrt{2} \\ 3\sqrt{5}(5 - 3\sqrt{2}) &= \underline{15\sqrt{5} - 9\sqrt{10}} \\ & 10 - 6\sqrt{2} + 15\sqrt{5} - 9\sqrt{10} = 10 \\ & + 3(5\sqrt{5} - 2\sqrt{2} - 3\sqrt{10}). \quad \text{Ans.} \end{aligned}$$

2.
$$\begin{aligned} &= 7(9 - 5\sqrt{2}) = 63 - 35\sqrt{2}; \\ 2\sqrt{32}(9 - 5\sqrt{2}) &= 18\sqrt{32} - 10\sqrt{64} = 18\sqrt{16 \times 2} \\ & - 10.8 = 18.4\sqrt{2} - 80 = 72\sqrt{2} - 80; \end{aligned}$$
$$72\sqrt{2} - 80 + 63 - 35\sqrt{2} = 37\sqrt{2} - 17. \quad \text{Ans.}$$

3.
$$\begin{aligned} &= a(a - \sqrt{b}) = a^2 - a\sqrt{b} \\ \sqrt{b}(a - \sqrt{b}) &= \underline{\qquad a\sqrt{b} - b} \\ & a^2 - b. \quad \text{Ans.} \end{aligned}$$

4.
$$\begin{aligned} &= a + b + c + d + 2\sqrt{ab} + 2\sqrt{ac} + 2\sqrt{bc} + 2\sqrt{ad} \\ & + 2\sqrt{bd} + 2\sqrt{cd} = a + b + c + d + 2(\sqrt{ab} + \sqrt{ac} \\ & + \sqrt{bc} + \sqrt{ad} + \sqrt{bd} + \sqrt{cd}). \end{aligned}$$

5.
$$\begin{aligned} &= m(m + 2n^{\frac{1}{3}}) = m^2 + 2mn^{\frac{1}{3}} \\ n^{\frac{1}{3}}(m + 2n^{\frac{1}{3}}) &= \underline{\qquad mn^{\frac{1}{3}} + 2n^{\frac{2}{3}}} \\ & m^2 + 3mn^{\frac{1}{3}} + 2n^{\frac{2}{3}}. \quad \text{Ans.} \end{aligned}$$

6.
$$\begin{aligned} &= a^{\frac{1}{2}}(a^{\frac{1}{2}} + a^{\frac{1}{3}}) = a + a^{\frac{5}{6}} \\ -a^{\frac{1}{3}}(a^{\frac{1}{2}} + a^{\frac{1}{3}}) &= \underline{\quad -a^{\frac{5}{6}} - a^{\frac{2}{3}}} \\ & a - a^{\frac{2}{3}}. \quad \text{Ans.} \end{aligned}$$

7. $= a^2 + 2aa^{-1} + a^{-2} = a^2 + 2 + a^{-2} = a^2 + 2 + \frac{1}{a^2}.$

8.
$$\begin{aligned} &= (a^{\frac{1}{2}})^4 - 4(a^{\frac{1}{2}})^3(a^{-\frac{1}{2}}) + 6(a^{\frac{1}{2}})^2(a^{-\frac{1}{2}})^2 - 4(a^{\frac{1}{2}})(a^{-\frac{1}{2}})^3 \\ & + (a^{-\frac{1}{2}})^4 \\ &= a^2 - 4a^{\frac{3}{2}-\frac{1}{2}} + 6a^{1-1} - 4a^{\frac{1}{2}-\frac{3}{2}} + a^{-2} = a^2 - 4a + 6 \\ & - \frac{4}{a} + \frac{1}{a^2}. \end{aligned}$$

9. $$\begin{array}{l} = a + b\sqrt{x+y} \\ \underline{a - b\sqrt{x+y}} \\ a^2 + ab\sqrt{x+y} \\ \underline{\quad - ab\sqrt{x+y} - b^2(x+y)} \\ a^2 - b^2(x+y) = a^2 - b^2x - b^2y. \end{array}$$

10. $$\begin{array}{l} m + n\sqrt{a+b} \\ \underline{m - n\sqrt{a-b}} \\ m^2 + mn\sqrt{a+b} \\ \underline{\quad - mn\sqrt{a-b} - n^2\sqrt{a^2-b^2}} \\ m^2 + mn\sqrt{a+b} - mn\sqrt{a-b} - n^2\sqrt{a^2-b^2}. \end{array}$$

11. $$\begin{array}{l} x + \sqrt{x^2-1} \\ \underline{x - \sqrt{x^2-1}} \\ x^2 + x\sqrt{x^2-1} \\ \underline{\quad - x\sqrt{x^2-1} - (x^2-1)} \\ x^2 - (x^2-1) = x^2 - x^2 + 1 = 1. \end{array}$$

12. $$\begin{array}{l} \sqrt{b^2+1} + b \\ \underline{\sqrt{b^2+1} - b} \\ b^2 + 1 + b\sqrt{b^2+1} \\ \underline{\quad - b\sqrt{b^2+1} - b^2} \\ b^2 + 1 - b^2 = 1. \end{array}$$

13. $= \sqrt{2}\,\sqrt{2} + \sqrt{2} = \sqrt{2}\,(\sqrt{2} + 1).$

14. $= 3.3^{\frac{1}{2}} + 2.3^{\frac{1}{2}} = 3^{\frac{1}{2}}(3 + 2) = 3^{\frac{1}{2}}.5.$

15. $= (a + b)(a + b)^{\frac{1}{2}}.$ **16.** $= y^{\frac{1}{2}}(1 + ay^2 - by^4)^{\frac{1}{2}}.$

17. $= \sqrt{x-y}\,\sqrt{x-y} - \sqrt{x-y} = \sqrt{x-y}\,(\sqrt{x-y} - 1).$

18. $$= \frac{\sqrt{2}\,\sqrt{2}}{\sqrt{2}} = \sqrt{2}.$$

19. $$= \frac{\sqrt{a+b}}{\sqrt{a+b}\,\sqrt{a+b}} = \frac{1}{\sqrt{a+b}}.$$

20. Divide num. and denom. by $x^{\frac{1}{2}} = \dfrac{ax + b}{ax - b}.$

21. Divide num. and denom. by $(a-x)^{\frac{1}{2}} = \dfrac{(a-x)^{\frac{1}{2}}+1}{(a-x)^{\frac{1}{2}}-1}$.

22. $= \dfrac{\sqrt{(a+b)(a-b)}}{\sqrt{(a+b)(a+b)}} = \dfrac{\sqrt{a+b}\sqrt{a-b}}{\sqrt{a+b}\sqrt{a+b}} = \dfrac{\sqrt{a-b}}{\sqrt{a+b}}$. Ans.

§ 185.

1. $\times$ by $\sqrt{a-6} = \dfrac{\sqrt{a^2-36}}{a-6}$.

2. $\times$ by $y^{\frac{1}{2}} = \dfrac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{y} = \dfrac{\sqrt{xy}}{y}$.

3. $\times$ by $\sqrt{1-x} = \dfrac{\sqrt{1-x^2}}{1-x}$.

4. $\times$ by $\sqrt{5} = \dfrac{7\sqrt{15}}{9.5} = \dfrac{7\sqrt{15}}{45}$.

5. $\times$ by $\sqrt{6} = \dfrac{2\sqrt{108}}{3.6} = \dfrac{\sqrt{108}}{9} = \dfrac{6\sqrt{3}}{9} = \dfrac{2\sqrt{3}}{3}$.

6. $\times$ by $\sqrt{2} = \dfrac{5\sqrt{48}}{2.2} = \dfrac{5\sqrt{48}}{4} = \dfrac{5.4\sqrt{3}}{4} = 5\sqrt{3}$.

7. $\times$ by $a+\sqrt{b} = \dfrac{(a+\sqrt{b})^2}{a^2-b}$.

8. $\times$ by $a-\sqrt{x} = \dfrac{(a-\sqrt{x})^2}{a^2-x}$.

9. $\times$ by $\sqrt{x}+\sqrt{y} = \dfrac{(\sqrt{x}+\sqrt{y})^2}{x-y}$.

10. $\times$ by $a-\sqrt{x+y} = \dfrac{a^2-a\sqrt{x+y}+2a\sqrt{x+y}-2(x+y)}{a^2-(x+y)}$

$$= \frac{a^2+a(x+y)^{\frac{1}{2}}-2(x+y)}{a^2-x-y}.$$

11. $\times$ by $\sqrt{5}+\sqrt{3}=\dfrac{2\sqrt{15}+7\sqrt{25}+2\sqrt{9}+7\sqrt{15}}{5-3}$

$$=\frac{9\sqrt{15}+7.5+2.3}{2}=\frac{9\sqrt{15}+41}{2}.$$

12. $\times$ by $\sqrt{x}-\sqrt{x+y}=\dfrac{(\sqrt{x}-\sqrt{x+y})^2}{x-x-y}$

$$=\frac{(\sqrt{x}-\sqrt{x+y})^2}{-y}.$$

13. $\times$ by $x+\sqrt{x^2-a^2}=\dfrac{x+\sqrt{x^2-a^2}}{x^2-(x^2-a^2)}=\dfrac{x+(x^2-a^2)^{\frac{1}{2}}}{a^2}.$

14. $\times$ by $a^{\frac{1}{2}}-(a+1)^{\frac{1}{2}}=\dfrac{a^{\frac{1}{2}}-(a+1)^{\frac{1}{2}}}{a-(a+1)}=\dfrac{a^{\frac{1}{2}}-(a+1)^{\frac{1}{2}}}{-1}$

$$=(a+1)^{\frac{1}{2}}-a^{\frac{1}{2}}.$$

15. $\times$ by $\sqrt{x+a}+\sqrt{x-a}=\dfrac{(\sqrt{x+a}+\sqrt{x-a})^2}{x+a-(x-a)}$

$$=\frac{x+a+2\sqrt{x^2-a^2}+x-a}{x+a-x+a}=\frac{2x+2\sqrt{x^2-a^2}}{2a}$$

$$=\frac{x+\sqrt{x^2-a^2}}{a}.$$

§ 186.

1. $=3+2.$ **2.** $=x+2.$ **3.** $=2x^2+\frac{1}{2}.$

4. Is not a square, as there is no double product and b^2 is negative.

5. $2a^n+3b^n.$

6. Is not a square, since b^2 is negative.

7. $x^2-\frac{1}{2}ay.$ **8.** $ab-cd.$ **9.** $m^{\frac{1}{2}}+n^{\frac{1}{2}}.$

10. Is not a square, as $2ax$ is not the double product of a and y, the roots of the square terms.

11. $a^{\frac{1}{2}}+2b^{\frac{1}{2}}.$

12. $a^{\frac{1}{2}}-a^{-\frac{1}{2}}$, or $a^{\frac{1}{2}}-\dfrac{1}{a^{\frac{1}{2}}}.$

13. $5p^2-3q.$

14. $h+3m^{2n}.$

15. $7xy-3z.$

16. $3m^{4n}-\dfrac{pq}{3}.$

§ 187.

1. $x^2 + 2xy$ needs y^2 for its completion, making then $x^2 + 2xy = (x+y)^2 - y^2$.

2. $x^2 + 4xy = (x + 2y)^2 - 4y^2$.

3. $x^2 + 6ax = (x + 3a)^2 - 9a^2$.

4. $4x^2 + 4xy = (2x + y)^2 - y^2$.

5. $4a^2 + 4ay = (2a + y)^2 - y^2$.

6. $9x^2 + ax = (3x + \frac{1}{6}a)^2 - \dfrac{a^2}{36}$.

7. $(4x + \frac{41}{4}m)^2 - (\frac{41}{4})^2 m^2$.

8. $(x+2)^2 - 4$.

9. $(ax + a)^2 - a^2$.

10. $\left(bx + \dfrac{1}{bx}\right)^2 - \dfrac{1}{b^2x^2}$

11. $\left(mx + \dfrac{1}{2mx}\right)^2 - \dfrac{1}{4m^2x^2}$.

12. $\left(3bx + \dfrac{1}{6}\right)^2 - \dfrac{1}{36}$

13. $\left(\dfrac{1}{2x} + x\right)^2 - x^2$.

14. $\left(\dfrac{1}{3ax} - 9a^2x\right)^2 - 81a^4x^2$.

§ 188.

1. $(m^{\frac{1}{2}} - n^{\frac{1}{2}})(m^{\frac{1}{2}} + n^{\frac{1}{2}})$.

2. $(m^{\frac{1}{2}} - 1)(m^{\frac{1}{2}} + 1)$.

3. $(a^{\frac{1}{2}}m^{\frac{1}{2}} - b^{\frac{1}{2}}n^{\frac{1}{2}})(a^{\frac{1}{2}}m^{\frac{1}{2}} + b^{\frac{1}{2}}n^{\frac{1}{2}})$.

4. $(2am^{\frac{1}{2}} - 3)(2am^{\frac{1}{2}} + 3)$.

5. $(x - m^{\frac{1}{2}})(x + m^{\frac{1}{2}})$.

6. $[x - (m+n)^{\frac{1}{2}}][x + (m+n)^{\frac{1}{2}}]$.

7. $[(x-a) - \frac{1}{2}(m-n)^{\frac{1}{2}}][(x-a) + \frac{1}{2}(m-n)^{\frac{1}{2}}]$.

8. $[x - (m-n)^{\frac{1}{2}}][x + (m-n)^{\frac{1}{2}}]$.

9. $[(a+b) - (4p^2 - q)^{\frac{1}{2}}][(a+b) + (4p^2 - q)^{\frac{1}{2}}]$,
or $\{a + b - [(2p - q^{\frac{1}{2}})(2p + q^{\frac{1}{2}})]^{\frac{1}{2}}\}$
$\{a + b + [(2p - q^{\frac{1}{2}})(2p + q^{\frac{1}{2}})]^{\frac{1}{2}}\}$.

10. $(x+y)^2 - (m+n)^3$
$= [x + y - (m+n)^{\frac{3}{2}}][x + y + (m+n)^{\frac{3}{2}}]$.

12. $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

13. $2 + \sqrt{3}$.

14. $3 - \sqrt{5}$.

15. $2a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

16. $(a+b)^{\frac{1}{2}} + x$.

17. $\sqrt{3} + \sqrt{5}$.

18. $\sqrt{3} - \sqrt{5}$.

19. $\frac{x^{\frac{1}{2}}}{2} - \frac{y^{\frac{1}{2}}}{2}$.

20. $a^{\frac{1}{2}} - 1$.

21. $a^{\frac{1}{2}} - a^{\frac{1}{2}}$.

22. $a^{\frac{1}{2}} + \frac{1}{a^{\frac{1}{2}}}$.

23. $a^{\frac{1}{2}} - \frac{a^{\frac{1}{2}}}{2}$.

24. $\frac{a^{\frac{1}{2}}}{2} + \frac{a^{\frac{1}{2}}}{3}$.

25. $\frac{a^{\frac{1}{2}}}{4} + \frac{1}{2}$.

26. $a^{\frac{1}{2}} + a^{-\frac{1}{2}}$.

27. $2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$.

28. $(a+b)^{\frac{1}{2}} - \frac{2}{(a+b)^{\frac{1}{2}}}$.

§ 190.

1. Cl. of fr., $p = qx^{\frac{1}{2}}$, $x^{\frac{1}{2}} = \frac{p}{q}$; square, $x = \frac{p^2}{q^2}$.

2. Cl. of fr., $x^{\frac{1}{3}}c = a + b$, $x^{\frac{1}{3}} = \frac{a+b}{c}$; cube, $x = \frac{(a+b)^3}{c^3}$.

3. Cl. of fr., $ax^{\frac{1}{3}} - a^2 = bx^{\frac{1}{3}} - b^2$; tr., $ax^{\frac{1}{3}} - bx^{\frac{1}{3}} = a^2 - b^2$,
or $(a-b)x^{\frac{1}{3}} = a^2 - b^2$; $x^{\frac{1}{3}} = \frac{a^2 - b^2}{a - b} = a + b$;
cube, $x = (a+b)^3$.

4. Cl. of fr., $216 = x^3$, $x = \sqrt[3]{216} = 6$.

5. Cl. of fr., $x^2 - 2ax - bx + 2ab = 2x^2 - 2ax - bx + ab$;
tr., $-2x^2 + x^2 - 2ax + 2ax - bx + bx = ab - 2ab$,
or $-x^2 = -ab$, $x^2 = ab$, $x = \sqrt{ab}$.

6. Cl. of fr., $x^4 - bx^2 - nax^2 + nab = nx^4 - nax^2 - bx^2 + ab$;
tr., $x^4 - nx^4 - bx^2 + bx^2 - nax^2 + nax^2 = ab - nab$,
or $x^4 - nx^4 = ab - nab$,
or $(1-n)x^4 = ab(1-n)$, $x^4 = ab$, $x = \sqrt[4]{ab}$.

7. Cl. of fr., $a^2 - b^2 = x^{\frac{m}{n} + \frac{p}{q}}$; or $x^{\frac{mq+np}{nq}} = a^2 - b^2$;
$x^{mq+np} = (a^2 - b^2)^{nq}$, $x = (a^2 - b^2)^{\frac{nq}{mq+np}}$. Ans.

8. Cl. of fr., $x = y$.

9. Cl. of fr., $\sqrt{x^2 - a^4} = b^2 - a^2$; square, $x^2 - a^4 =$
$b^4 - 2a^2b^2 + a^4$; tr., $x^2 = b^4 - 2a^2b^2 + 2a^4$,
$x = (b^4 - 2a^2b^2 + 2a^4)^{\frac{1}{2}}$.

10. Cube the equation, $x - a = b^3$; tr., $x = b^3 + a$.

11. Square the equation, $x^2 - a^2 = m^2x^2$; tr., $x^2 - m^2x^2 = a^2$,
or $(1 - m^2)\,x^2 = a^2$, $x^2 = \dfrac{a^2}{1 - m^2}$, $x = \dfrac{a}{(1 - m^2)^{\frac{1}{2}}}$.

12. Square, $\sqrt{x} - \sqrt{b} = n^2x^{\frac{1}{2}}$; tr., $x^{\frac{1}{2}} - n^2x^{\frac{1}{2}} = b^{\frac{1}{2}}$, or
$(1 - n^2, x^{\frac{1}{2}} = b^{\frac{1}{2}}$; square, $(1 - n^2)^2\,x = b$, $x = \dfrac{b}{(1 - n^2)^2}$.

§ 191.

1. Let x be the 1st, $2x$ the 2d, $\dfrac{2x}{3}$ the 3d; then

$$x^2 + 4x^2 + \frac{4x^2}{9} = 196.$$

Cl. of fr., $9x^2 + 36x^2 + 4x^2 = 1764$, or $49x^2 = 1764$;
extract $\sqrt{\ }$, $7x = \pm 42$, $x = \pm 6$, $2x = \pm 12$, $\dfrac{2x}{3} = \pm 4$.

2. Let x^2 be one, and $x^2 + 81$ be the square of the other; then $2x^2 + 81 = 369$, or $2x^2 = 369 - 81 = 288$; $x^2 = 144$, $x = \pm 12$; $x^2 + 81 = 225$, the square of the second, or $\sqrt{225} = \pm 15$, the second.

3. Let x be the mean of the length and breadth; then $x + 6 =$ the length, and $x - 6$ the breadth, and $(x + 6)(x - 6) = 1645$, or $x^2 - 36 = 1645$, $x^2 = 1681$, $x = \pm 41$, $x + 6 = 47$ or -35, and $x - 6 = 35$ or -47.

4. Let x be the number; then $(x + 9)(x - 9) = 175$, $x^2 - 81 = 175$, or $x^2 = 256$, $x = \pm 16$, $x + 9 = 25$ or -7,
$x - 9 = 7$ or -25.

5. $(x + a)(x - a) = 2a + 1$, $x^2 - a^2 = 2a + 1$;
tr., $x^2 = a^2 + 2a + 1$; extract $\sqrt{\ }$, $x = \pm (a + 1)$.

6. $x =$ 1st, and $2x =$ 2d; $4x^2 - x^2 = 192$, $3x^2 = 192$,
$x^2 = 64$, $x = \pm 8$, $2x = \pm 16$.

7. $x = $ 1st, and $8x = $ 2d; $\sqrt[3]{8x} + \sqrt[3]{x} = 12$; that is,
$2\sqrt[3]{x} + \sqrt[3]{x} = 12$, or $3\sqrt[3]{x} = 12$, $\sqrt[3]{x} = 4$, $x = 64$,
$8x = 512$.

8. $x = $ 1st, $3x = $ 2d, $\sqrt{4x} \times x = 128$, or $2\sqrt{x^3} = 128$,
$\sqrt{x^3} = 64$; square, $x^3 = 4096$, $x = 16$, $3x = 48$.

9. Let $2x$ be one, and $3x$ the other; then
$4x^2 + 9x^2 = 325$, or $13x^2 = 325$, $x^2 = 25$, $x = \pm 5$.
$2x = 10$, $3x = 15$.

10. Let mx be one, and nx the other;
$mx + nx = (mx - nx)^2$, or $mx + nx = m^2x^2 - 2mnx^2 + n^2x^2$;
$\div x$, $m + n = m^2x - 2mnx + n^2x$, or $(m^2 - 2mn + n^2)x$,

$$x = \frac{m+n}{m^2 - 2mn + n^2},\quad mx = \frac{m^2 + mn}{(m-n)^2}.$$

11. Let $9x$ be one, and $7x$ the other;
$(9x - 7x)^{\frac{1}{3}}(9x + 7x)^{\frac{1}{2}} = 16$,
or $2^{\frac{1}{3}}x^{\frac{1}{3}} \times 16^{\frac{1}{2}}x^{\frac{1}{2}} = 16$, or $2^{\frac{1}{3}}x^{\frac{1}{3}} \times 4x^{\frac{1}{2}} = 16$, $4.2^{\frac{1}{3}}x^{\frac{5}{6}} = 16$,

$$x^{\frac{5}{6}} = \frac{4}{2^{\frac{1}{3}}} = 2^{\frac{5}{3}}.$$

Raise to 6th power, $x^5 = 2^{10}$.
Extracting 5th root, $x = 2^2 = 4$.
Therefore the numbers are 36 and 28.

12. $\times$ (1) by a', $a'ax^2 + a'by^2 = a'c$ (3);
$\times$ (2) by a, $a'ax^2 + ab'y^2 = ac'$ (4);
(3) $-$ (4), $a'by^2 - ab'y^2 = a'c - ac'$;
$(a'b - ab')y^2 = a'c - ac'$;

$$y^2 = \frac{a'c - ac'}{a'b - ab'},\quad y = \left(\frac{a'c - ac'}{a'b - ab'}\right)^{\frac{1}{2}}.$$

$\times$ (1) by b', $ab'x^2 + bb'y^2 = b'c$ (5);
$\times$ (2) by b, $a'bx^2 + bb'y^2 = bc'$ (6):
(5) $-$ (6), $ab'x^2 - a'bx^2 = b'c - bc'$;
$(ab' - a'b)x^2 = b'c - bc'$;

$$x^2 = \frac{b'c - bc'}{ab' - a'b},\quad x = \left(\frac{b'c - bc'}{ab' - a'b}\right)^{\frac{1}{2}}.$$

13. Let x be the amount by which each side differs from half their sum; then
$17 + x = $ one side, and $17 - x$ the other;

$(17+x)^2+(17-x)^2=26^2$, or $x^2+34x+289+x^2-34x+289=676$, $2x^2=676-578$ or 98, $x^2=49$, $x=7$, $17+x=24$, and $17-x=10$.

Therefore the sides are 10 and 24.

14. Let x be the required time; then one has travelled mx feet and the other nx feet;

$$m^2x^2+n^2x^2=c^2,\qquad (m^2+n^2)x^2=c^2,$$
$$x^2=\frac{c^2}{m^2+n^2},\qquad x=\frac{c}{(m^2+n^2)^{\frac{1}{2}}}.$$

15. Let t be the time; then $t^2\,16=h$, $\quad t^2=\frac{h}{16}$, $t=\frac{\sqrt{h}}{4}$.

§ 195.

1. Reducing first member to common denominator,

$$\frac{x^2+4x+4-(x^2-4x+4)}{x^2-4}=\frac{5}{6},$$
$$\frac{x^2+4x+4-x^2+4x-4}{x^2-4}=\frac{5}{6},$$
$$\frac{8x}{x^2-4}=\frac{5}{6}.$$

Cl. of fr., $48x=5x^2-20$;

transpose, $5x^2-48x=20$,

$$x^2-\tfrac{48}{5}x=4,$$
$$x^2-\tfrac{48}{5}x+(\tfrac{24}{5})^2=4+(\tfrac{24}{5})^2=\tfrac{676}{25},$$
$$x-\tfrac{24}{5}=\pm\tfrac{26}{5},$$
$$x=\tfrac{50}{5}\text{ or }-\tfrac{2}{5},$$
$$x=10\text{ or }-\tfrac{2}{5}.$$

2. Reducing the first member,

$$\frac{y^2+8y+16+y^2-8y+16}{y^2-16}=\frac{10}{3},\quad \frac{2y^2+32}{y^2-16}=\frac{10}{3}.$$

Cl. of fractions, $6y^2+96=10y^2-160$;

transpose, $-4y^2=-256$,

$$y^2=64,$$
$$y=\pm 8.$$

3. Cl. of fr., $\quad 3x-6+6x-6=4x^2-12x+8$;

transpose, $\quad -4x^2+12x+9x=8+12$,

or $4x^2-21x=-20$;

complete sq., $4x^2-21x+\frac{441}{16}=-20+\frac{441}{16}=\frac{121}{16}$;

extract $\sqrt{\ }$, $\quad 2x-\frac{21}{4}=\pm\frac{11}{4}$,

$$x=\tfrac{21}{8}\pm\tfrac{11}{8}=4\text{ or }1\tfrac{1}{4}.$$

4. Transpose, $y^2 - 2ay = b^2 - a^2$;
complete, $y^2 - 2ay + a^2 = b^2 - a^2 + a^2 = b^2$;
extract $\sqrt{\ }$, $y - a = \pm b$,
$$y = a \pm b.$$

5. Clear second member of fractions,
$$\frac{1}{a+b+x} = \frac{bx+ax+ab}{abx}.$$
Cl. of fr., $abx = abx + a^2x + a^2b + b^2x + abx + ab^2 + bx^2 + ax^2 + abx$;
tr., $-ax^2 - bx^2 + abx - abx - abx - abx - b^2x - a^2x = a^2b + ab^2$;
unite and change signs, $(a+b)x^2 + (a^2 + 2ab + b^2)x = -a^2b - ab^2 = -ab(a+b)$;
$\div (a+b) = x^2 + (a+b)x = -ab$;
complete, $x^2 + (a+b)x + \dfrac{(a+b)^2}{4} = -ab + \dfrac{a^2+2ab+b^2}{4} = \dfrac{a^2-2ab+b^2}{4}$;
extract $\sqrt{\ }$, $x + \dfrac{a+b}{2} = \pm\dfrac{a-b}{2}$,
$$x = \pm\frac{a-b}{2} - \frac{a+b}{2} = -b \text{ or } -a.$$

6. Clear of fractions, $x^2 - a^2 = $ L.C.M.;
$$a^2 + bx - ba + bx + ba = x^2 - a^2;$$
$$x^2 - 2bx = 2a^2,$$
$$x^2 - 2bx + b^2 = 2a^2 + b^2,$$
$$x = b \pm \sqrt{2a^2 + b^2}.$$

7. Multiply num. and denom. by $x^2 - a^2$
$$= \frac{x^2 - a^2 + x^2 + 2ax + a^2}{x^2 - a^2 - (x^2 - 2ax + a^2)} = 3;$$
$$\frac{2x^2 + 2ax}{-2a^2 + 2ax} = 3.$$
Cl. of fr., $2x^2 + 2ax = -6a^2 + 6ax$;
transpose, $2x^2 - 4ax = -6a^2$,
$$x^2 - 2ax = -3a^2;$$
complete, $x^2 - 2ax + a^2 = -3a^2 + a^2 = -2a^2$;
extract $\sqrt{\ }$, $x - a = \pm\sqrt{-2a^2} = \pm a\sqrt{-2}$,
$$x = a \pm a\sqrt{-2} = a(1 \pm \sqrt{-2}).$$

8. First change the sign of the num. and denom. of second fraction, making the expression

$$\frac{2}{2+y} - \frac{-y}{4-y^2} + \frac{2}{2-y} = 4;$$

$$4 - y^2 = \text{L.C.M.};$$

then $4 - 2y - (-y) + 4 + 2y = 16 - 4y^2$,

or $4 - 2y + y + 4 + 2y = 16 - 4y^2$;

transpose, $4y^2 + y = 8$;

complete, $4y^2 + y + \frac{1}{16} = \frac{129}{16}$;

extract $\sqrt{\ }$, $2y + \frac{1}{4} = \pm\sqrt{\frac{129}{16}} = \pm\frac{1}{4}\sqrt{129}$;

$$y = -\tfrac{1}{8} \pm \tfrac{1}{8}\sqrt{129} = \tfrac{1}{8}(-1 \pm \sqrt{129}).$$

9. Clear of fractions, 1st member =

$$\frac{y^2 + 2ay + a^2 - (y^2 - 2ay + a^2)}{y^2 - a^2} = \frac{4ay}{y^2 - a^2};$$

$$\frac{1}{y-a} - \frac{1}{y^2 - a^2} + \frac{1}{y-a} = \frac{y + a - 1 + y + a}{y^2 - a^2}$$

$$= \frac{2y + 2a - 1}{y^2 - a^2};$$

then $\dfrac{4ay}{y^2 - a^2} = \dfrac{2y + 2a - 1}{y^2 - a^2}$,

$$4ay = 2y + 2a - 1;$$

transpose, $4ay - 2y = 2a - 1$,

$$(4a - 2)y = 2a - 1,$$

$$y = \frac{2a - 1}{4a - 2} = \frac{1}{2}.$$

10. Cl. of fr., $ax - x^2 - (ax + x^2) + 3a^2 - 3x^2 = 0$,

$$ax - x^2 - ax - x^2 + 3a^2 - 3x^2 = 0;$$

transpose, $-5x^2 = -3a^2$,

$$x^2 = \frac{3a^2}{5};$$

$$x = \pm\sqrt{\frac{3a^2}{5}} = \pm a\sqrt{\frac{3}{5}}.$$

§195.—Problems.

1. Let x = one,
$x + 6$ = the other;
$$\text{then } x(x+6) = 567,$$
$$x^2 + 6x = 567;$$
complete, $x^2 + 6x + 9 = 567 + 9 = 576;$
extract $\surd$, $x + 3 = \pm 24,$
$$x = \pm 24 - 3 = 21 \text{ or } -27,$$
$$x + 6 = 27 \text{ or } -21.$$

2. Let x = one,
$x + 6$ = the other;
$$\text{then } (x+6)^3 - x^3 = 936,$$
$$x^3 + 18x^2 + 108x + 216 - x^3 = 936;$$
tr., $18x^2 + 108x = 936 - 216 = 720,$
$$x^2 + 6x = 40;$$
complete, $x^2 + 6x + 9 = 40 + 9 = 49;$
extract $\surd$, $x + 3 = \pm 7,$
$$x = -3 \pm 7 = 4 \text{ or } -10,$$
$$x + 6 = 10 \text{ or } -4.$$

3. Let x = one part,
$34 - x$ = the other;
$$\text{then } (34-x)^2 + x^2 = 2(34-x)x,$$
$$\text{or } 1156 - 68x + x^2 + x^2 = 68x - 2x^2;$$
tr., $2x^2 + 2x^2 - 68x - 68x = -1156,$
$$4x^2 - 136x = -1156,$$
$$x^2 - 34x = -289;$$
complete, $x^2 - 34x + 17^2 = -289 + 289 = 0;$
extract $\surd$, $x - 17 = 0,$
$$x = 17,$$
$$34 - x = 17.$$

4. Let x = one number,
$60 - x$ = the other;
$$(60-x)^2 + x^2 = 1872,$$
$$3600 - 120x + x^2 + x^2 = 1872;$$
transpose, $2x^2 - 120x = 1872 - 3600 = -1728,$
$$x^2 - 60x = -864;$$
complete, $x^2 - 60x + 30^2 = 900 - 864 = 36;$
extract $\surd$, $x - 30 = \pm 6,$
$$x = 30 \pm 6 = 36 \text{ or } 24,$$
$$60 - x = 24 \text{ or } 36.$$

5.

$$\begin{aligned}
\text{Let } x &= \text{the first,}\\
x+5 &= \text{the second,}\\
2(x+5) &= \text{the third;}
\end{aligned}$$

$$x^2 + (x+5)^2 + (2x+10)^2 = 1225,$$

$$x^2 + x^2 + 10x + 25 + 4x^2 + 40x + 100 = 1225;$$

transpose, $6x^2 + 50x = 1225 - 125 = 1100;$

$\div 6,$ $x^2 + \frac{25}{3}x = \frac{550}{3};$

complete, $x^2 + \frac{25}{3}x + (\frac{25}{6})^2 = \frac{625}{36} + \frac{550}{3} = \frac{7225}{36};$

extract $\sqrt{\ }$, $x + \frac{25}{6} = \pm\frac{85}{6},$

$$x = \pm\tfrac{85}{6} - \tfrac{25}{6} = 10 \text{ or } -\tfrac{55}{3},$$

$$x + 5 = 15 \text{ or } -\tfrac{40}{3},$$

$$2(x+5) = 30 \text{ or } -\tfrac{80}{3}.$$

6.

$$\begin{aligned}
\text{Let } x &= \text{second,}\\
x-4 &= \text{first,}\\
x+4 &= \text{third,}\\
x+8 &= \text{fourth;}
\end{aligned}$$

then $x(x-4) + (x+4)(x+8) = 312,$

$$x^2 - 4x + x^2 + 12x + 32 = 312;$$

transpose, $2x^2 + 8x = 312 - 32 = 280;$

$\div 2,$ $x^2 + 4x = 140;$

complete, $x^2 + 4x + 4 = 140 + 4 = 144;$

extract $\sqrt{\ }$, $x + 2 = \pm 12,$

$$\begin{aligned}
x &= \pm 12 - 2 = 10 \text{ or } -14, && \text{II.}\\
x - 4 &= 6 \text{ or } -18, && \text{I.}\\
x + 4 &= 14 \text{ or } -10, && \text{III.}\\
x + 8 &= 18 \text{ or } -6. && \text{IV.}
\end{aligned}$$

7. Let x = number of pairs. Then $\frac{210}{x}$ = price per pair. $x - 5$ would be the number if there had been 5 less; then $\frac{210}{x-5}$ would have been the price, but this price would have been \$1 per pair more.

That is, $\frac{210}{x-5} = \frac{210}{x} + 1.$

Cl. of fr., $210x = 210x - 1050 + x^2 - 5x;$

transpose, $x^2 - 5x = 1050;$

complete, $x^2 - 5x + \frac{25}{4} = 1050 + \frac{25}{4} = \frac{4225}{4};$

$$x - \tfrac{5}{2} = \pm\tfrac{65}{2},$$

$x = \frac{5}{2} \pm \frac{65}{2} = 35$ or $-30.$ $\frac{210}{x} = \$6$ per pair.

8. Let $x =$ number of turkeys,

$x + 4 =$ " " chickens;

$\$10$ or $\dfrac{1000 \text{ cts.}}{x+4} =$ cost of each chicken,

$\$15.75$ or $\dfrac{1575 \text{ cts.}}{x} =$ " " " turkey;

$$\text{then } \frac{1575}{x} - \frac{1000}{x+4} = 35;$$

$$\div 5, \frac{315}{x} - \frac{200}{x+4} = 7;$$

Clear of fractions, $315x + 1260 - 200x = 7x^2 + 28x$;

trans., $-7x^2 + 315x - 200x - 28x = -1260$,

$-7x^2 + 87x = -1260$;

$\div -7$, $x^2 - \frac{87}{7}x = \frac{1260}{7}$;

complete, $x^2 - \frac{87}{7}x + (\frac{87}{14})^2 = \frac{1260}{7} + \frac{7569}{196} = \frac{42849}{196}$;

extract $\sqrt{\ }$, $x - \frac{87}{14} = \pm \frac{207}{14}$,

$x = \frac{87}{14} \pm \frac{207}{14} = 21$ or $-8\frac{8}{14}$,

$x + 4 = 25$.

As the negative root gives a fractional and negative result, it in this case is not a possible answer.

9. Let $x =$ number of sheep,

$\dfrac{240}{x} =$ price of each;

$x + 3 =$ number if there had been 3 more,

$\dfrac{240}{x+3} =$ price " " " " " "

$$\text{then } \frac{240}{x} - \frac{240}{x+3} = 4;$$

cl. of fr., $240x + 720 - 240x = 4x^2 + 12x$;

trans., $-4x^2 + 240x - 240x - 12x = -720$,

$-4x^2 - 12x = -720$;

$\div -4$, $x^2 + 3x = 180$;

complete, $x^2 + 3x + (\frac{3}{2})^2 = 180 + \frac{9}{4} = \frac{729}{4}$;

extract $\sqrt{\ }$, $x + \frac{3}{2} = \pm \frac{27}{2}$,

$x = \pm \frac{27}{2} - \frac{3}{2} = 12$ or -15.

As it is impossible for one to sell -15 sheep, we might interpret the negative result by considering that he *purchased* 15 sheep.

10. Let x be the number; then $\dfrac{960}{x} =$ price for each; after 2 left $\dfrac{960}{x-2} =$ price for each; and as this was 24 cents

more than the expected cost, $\frac{960}{x-2} = \frac{960}{x} + 24.$
Cl. of fr., $960x = 960x - 1920 + 24x^2 - 48x$;
eliminate and transpose, $24x^2 - 48x = 1920$;
$\div 24$, $x^2 - 2x = 80$;
comptete, $x^2 - 2x + 1 = 81$;
extract $\sqrt{}$, $x - 1 = \pm 9$, $x = 10$ or -11. Ans.

11. Let $x =$ the number bought;
$\frac{1000}{x} =$ the price of each.

After taking away the 30 rotten ones he had $x - 30$ left. If he gained \$3.20, he sold them for \$13.20 or $\frac{1320}{x-30}$ each; and as he sold them at an advance of 2 cents each, $\frac{1320}{x-30} - \frac{1000}{x} = 2.$
Cl. of fr., $1320x - 1000x + 30\,000 = 2x^2 - 60x$;
trans., $-2x^2 + 380x = -30\,000$;
$\div -2$, $x^2 - 190x = 15\,000$;
complete, $x^2 - 190x + (95)^2 = 15\,000 + 9025 = 24\,025$;
extract $\sqrt{}$, $x - 95 = \pm 155$,
$x = 95 \pm 155 = 250$ or -60.
The negative value can be understood as in example 9.

12. Let r be the rate.
$\frac{48}{r} =$ the time,
$\frac{48}{r+1} = \frac{48}{r} - 4;$
Clear of fractions, $48r = 48r + 48 - 4r^2 - 4r$;
transpose and eliminate, $4r^2 + 4r = 48$,
$r^2 + r = 12$;
complete, $r^2 + r + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$;
extract $\sqrt{}$, $r + \frac{1}{2} = \pm \frac{7}{2}$, $r = 3$ or -4.

13. If the whole perimeter is $160\ M$, one end and one side will be one half of $160\ M = 80\ M$.
Then let $x =$ the width,
$80 - x =$ the length,
$x(80 - x) =$ whole area $= 1575$.
$80x - x^2 = 1575$,
$x^2 - 80x = -1575$;
complete, $x^2 - 80x + (40)^2 = 1600 - 1575 = 25$.
extract $\sqrt{}$, $x - 40 = \pm 5$,
$x = 40 \pm 5 = 45$ or 35,
$80 - x = 35$ or 45.

14. Let $x =$ the breadth,

$x + a =$ the length.

$x(x+a) = m^2,$

$x^2 + ax = m^2;.$

complete, $x^2 + ax + \frac{a^2}{4} = m^2 + \frac{a^2}{4} = \frac{4m^2+a^2}{4};$

extract $\sqrt{\ }$, $x + \frac{a}{2} = \pm\sqrt{\frac{4m^2+a^2}{4}} = \pm\frac{1}{2}\sqrt{4m^2+a^2},$

$$x = -\frac{a}{2} \pm \frac{1}{2}\sqrt{4m^2+a^2} = \frac{1}{2}(-a \pm \sqrt{4m^2+a^2}),$$

$$x + a = a - \frac{a}{2} \pm \frac{1}{2}\sqrt{4m^2+a^2} = \frac{a}{2} \pm \frac{1}{2}\sqrt{4m^2+a^2},$$

$$x + a = \frac{1}{2}(a \pm \sqrt{4m^2+a^2}).$$

15. Let $x =$ the number of miles per hour the 2d travelled;
then $x =$ " " " hours " " "

$x^2 =$ " distance the second travelled.

As the first started 3 hours in advance, he travelled $x + 3$ hours.

$x + 3$ hours at 8 miles per hour $= 8(x + 3)$.

Now if the second travelled at the rate of x miles per hour, and by the conditions of the problem 18 hours were necessary to make the trip, the whole distance must be $18x$.

$\therefore x^2 + 8(x+3) =$ the whole distance, or $18x$;

$x^2 + 8x + 24 = 18x;$

transpose, $x^2 - 10x = -24;$

complete, $x^2 - 10x + 25 = 25 - 24 = 1;$

extract $\sqrt{\ }$, $x - 5 = \pm 1,$

$x = 5 \pm 1 = 6$ or $4,$

$x + 3 = 9$ or $7;$

6 hours at 6 miles per hour $=$ 36 miles;
9 " " 8 " " " $=$ 72 "
Whole distance $=$ 108 "
or 4 hours at 4 miles per hour $=$ 16 miles;
7 " " 8 " " " $=$ 56 "

$\therefore$ whole distance $= 108$ or 72 miles.

§ 196.

1. Let $x =$ the number;

then $x + 2x^{\frac{1}{2}} = 99;$

complete, $x + 2x^{\frac{1}{2}} + 1 = 99 + 1 = 100;$

extract $\sqrt{\ }$, $x^{\frac{1}{2}} + 1 = \pm 10,$

$x^{\frac{1}{2}} = -1 \pm 10 = 9$ or $-11;$

square, $x = 81$ or $121.$

2. Let $x =$ the number;
then $x - 2x^{\frac{1}{2}} = 99$;
complete, $x - 2x^{\frac{1}{2}} + 1 = 100$;
extract $\sqrt{\ }$, $x^{\frac{1}{2}} - 1 = \pm 10$,
$x^{\frac{1}{2}} = 1 \pm 10 = 11$ or -9;
square, $x = 121$ or 81.

3. Let $x =$ the number;
then $\dfrac{x}{5} - x^{\frac{1}{2}} = 30$.
Clear of fractions, $x - 5x^{\frac{1}{2}} = 150$;
complete, $x - 5x^{\frac{1}{2}} + \frac{25}{4} = 150 + \frac{25}{4} = \frac{625}{4}$;
extract $\sqrt{\ }$, $x^{\frac{1}{2}} - \frac{5}{2} = \pm \frac{25}{2}$,
$x^{\frac{1}{2}} = \frac{5}{2} \pm \frac{25}{2} = 15$ or -10;
square, $x = 225$ or 100.

4. $x + x^{\frac{1}{2}} = 306$;
complete, $x + x^{\frac{1}{2}} + \frac{1}{4} = 306 + \frac{1}{4} = \frac{1225}{4}$;
extract $\sqrt{\ }$, $x^{\frac{1}{2}} + \frac{1}{2} = \pm \frac{35}{2}$,
$x^{\frac{1}{2}} = -\frac{1}{2} \pm \frac{35}{2} = 17$ or -18;
square, $x = 289$ or 324.

5. Let $x =$ the number.
$\dfrac{3x - 10x^{\frac{1}{2}} - 96}{x} = 2.$
Clear of fractions, $3x - 10x^{\frac{1}{2}} - 96 = 2x$;
transpose, $x - 10x^{\frac{1}{2}} = 96$;
complete, $x - 10x^{\frac{1}{2}} + 25 = 96 + 25 = 121$;
extract $\sqrt{\ }$, $x^{\frac{1}{2}} - 5 = \pm 11$,
$x^{\frac{1}{2}} = 5 \pm 11 = 16$ or -6;
square, $x = 256$ or 36.

6. Clear of fractions, $y^4 - 6y^2 = 45$;
complete, $y^4 - 6y^2 + 9 = 45 + 9 = 54$;
extract $\sqrt{\ }$, $y^2 - 3 = \pm\sqrt{54} = \pm 3\sqrt{6}$,
$y^2 = 3 \pm 3\sqrt{6} = 3(1 \pm \sqrt{6})$,
$y = \sqrt{3(1 \pm \sqrt{6})}$.

7. $\div 3$, $y^4 - \frac{7}{3}y^2 = \frac{25}{3}$;
complete, $y^4 - \frac{7}{3}y^2 + (\frac{7}{6})^2 = \frac{25}{3} + \frac{49}{36} = \frac{349}{36}$;
extract $\sqrt{\ }$, $y^2 - \frac{7}{6} = \pm\sqrt{\frac{349}{36}} = \pm \frac{1}{6}\sqrt{349}$,
$y^2 = \frac{7}{6} \pm \frac{1}{6}\sqrt{349} = \frac{1}{6}(7 \pm \sqrt{349})$,
$y = \sqrt{\frac{1}{6}(7 \pm \sqrt{349})}$.

8. $\div 5,\ y^{\frac{1}{2}} - \frac{8}{5}y^{\frac{1}{4}} = \frac{13}{5};$

complete, $y^{\frac{1}{2}} - \frac{8}{5}y^{\frac{1}{4}} + \frac{16}{25} = \frac{13}{5} + \frac{16}{25} = \frac{81}{25};$

extract $\sqrt{\ }$, $y^{\frac{1}{4}} - \frac{4}{5} = \pm \frac{9}{5},$

$y^{\frac{1}{4}} = \frac{4}{5} \pm \frac{9}{5} = \frac{13}{5}$ or $-1;$

raise to 4th power, $y = (\frac{13}{5})^4$ or $1.$

9. For $x^2 + a^2$ substitute y.

Then $y^{\frac{m}{n}} - 4y^{\frac{m}{2n}} = a^2 - 2 + \dfrac{1}{a^2};$

complete, $y^{\frac{m}{n}} - 4y^{\frac{m}{2n}} + 4 = a^2 - 2 + \dfrac{1}{a^2} + 4 = a^2 + 2 + \dfrac{1}{a^2};$

extract $\sqrt{\ }$, $y^{\frac{m}{2n}} - 2 = a + \dfrac{1}{a},$

$y^{\frac{m}{2n}} = 2 + a + \dfrac{1}{a},$

or $(x^2 + a^2)^{\frac{m}{2n}} = 2 + a + \dfrac{1}{a};$

raise to $2n$th power, $(x^2 + a^2)^m = \left(2 + a + \dfrac{1}{a}\right)^{2n};$

extract mth root, $x^2 + a^2 = \left(2 + a + \dfrac{1}{a}\right)^{\frac{2n}{m}};$

transpose, $x^2 = \left(2 + a + \dfrac{1}{a}\right)^{\frac{2n}{m}} - a^2,$

$$x = \sqrt{\left(2 + a + \frac{1}{a}\right)^{\frac{2n}{m}} - a^2}.$$

§ 197.

1. By adding 2 we get $x^2 + 2 + \dfrac{1}{x^2} = \dfrac{17}{4} + 2 = \dfrac{25}{4};$

extract $\sqrt{\ }$, $x + \dfrac{1}{x} = \pm \dfrac{5}{2}.$

By taking the upper sign and clearing of fractions,

$2x^2 + 2 = 5x$, or $x^2 - \frac{5}{2}x = -1;$

complete, $x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{25}{16} - 1 = \frac{9}{16};$

extract $\sqrt{\ }$, $x - \frac{5}{4} = \pm \frac{3}{4}$, $x = \frac{5}{4} \pm \frac{3}{4} = 2$ or $\frac{1}{2}.$

By taking the lower sign and clearing of fractions,

$2x^2 + 2 = -5x$, or $x + \frac{5}{2}x = -1;$

complete, $x^2 + \frac{5}{2}x + \frac{25}{16} = -1 + \frac{25}{16} = \frac{9}{16};$

extract $\sqrt{\ }$, $x + \frac{5}{4} = \pm \frac{3}{4}$, $x = -\frac{5}{4} \pm \frac{3}{4} = -\frac{1}{2}$ or $-2.$

2. Add 2, $a^2x^2 + 2 + \frac{1}{a^2x^2} = m^2$;

extract $\sqrt{\ }$, $\quad ax + \frac{1}{ax} = \pm m$.

Take first sign and clear of fractions, $a^2x^2 + 1 = max$;

tr. and complete, $a^2x^2 - max + \frac{m^2}{4} = \frac{m^2}{4} - 1$;

extract $\sqrt{\ }$, $\quad ax - \frac{m}{2} = \pm \frac{\sqrt{m^2 - 4}}{2}$,

$$x = \frac{m}{2a} \pm \frac{\sqrt{m^2 - 4}}{2a}.$$

By taking the lower sign we would get

$$x = -\frac{m}{2a} \pm \frac{\sqrt{m^2 - 4}}{2a}.$$

3. Add 8, $\quad 16y^2 + 8 + \frac{1}{y^2} = 36$;

extract $\sqrt{\ }$, $\quad 4y + \frac{1}{y} = \pm 6$.

By taking the first sign, $4y^2 + 1 = 6y$;

tr. and complete, $y^2 - \frac{6y}{4} + \frac{9}{16} = \frac{9}{16} - \frac{1}{4} = \frac{5}{16}$;

extract $\sqrt{\ }$, $y - \frac{3}{4} = \pm \frac{\sqrt{5}}{4}$, $y = \frac{3}{4} \pm \frac{\sqrt{5}}{4}$.

By taking the other sign we would get

$$y = \pm \frac{\sqrt{5}}{4} - \frac{3}{4}.$$

4. Add $2m^2$, $\frac{m^4}{y^2} + 2m^2 + y^2 = 4m^2$;

extract $\sqrt{\ }$, $\quad \frac{m^2}{y} + y = \pm 2m$.

Take the first sign and cl. of fractions, $m^2 + y^2 = 2my$;
complete, $y^2 - 2my + m^2 = m^2 - m^2 = 0$;
extract $\sqrt{\ }$, $\quad y - m = \pm 0$,

$$y = m.$$

Take the other sign,

$$m^2 + y^2 = -2my,$$
$$y^2 + 2my + m^2 = m^2 - m^2 = 0,$$
$$y + m = 0,$$
$$y = -m.$$

5. If r is a root of the equation, $-r$ will also be, since the square root of a quantity is $\pm$.

In the equation $x^2 + \frac{1}{x^2} = a$, if we change x^2 into $\frac{1}{x^2}$, the expression remains unaltered; therefore the reciprocal of the root $\pm r$ is also a root of the same equation, or $-\frac{1}{r}$ and $\frac{1}{r}$.

§ 199.

1. $(x-1)(x+1) = x^2 - 1 = 0.$

2. $(x-3)(x-2) = x^2 - 5x + 6 = 0.$

3. $(x+3)(x+2) = x^2 + 5x + 6 = 0.$

4. $(x - 3 - 2\sqrt{10})(x - 3 + 2\sqrt{10}) = x^2 - 6x - 31 = 0.$

5. $(x - 7 - 2\sqrt{3})(x + 7 - 2\sqrt{3}) = x^2 - 14x + 37 = 0.$

6. $(x-1)(x-2) = x^2 - 3x + 2 = 0.$

7. $(x+1)(x-2) = x^2 - x - 2 = 0.$

8. $(x+1)(x+2) = x^2 + 3x + 2 = 0.$

9. $(x-1)(x+2) = x^2 + x - 2 = 0.$

10. $(x - 2 - \sqrt{5})(x - 2 + \sqrt{5}) = x^2 - 4x + 1 = 0.$

11. $(x - \frac{3}{4})(x - \frac{4}{5}) = x^2 - \frac{31}{20}x + \frac{3}{5} = 0.$

12. $(x - \frac{7}{2})(x - \frac{9}{2}) = x^2 - 8x + \frac{63}{4} = 0.$

13. $(x - 2 - \sqrt{2})(x - 2 + \sqrt{2}) = x^2 - 4x + 2 = 0.$

14. $(x - 9 - 2\sqrt{2})(x - 9 + 2\sqrt{2}) = x^2 - 18x + 73 = 0.$

15. $(x - 5 - 7\sqrt{5})(x - 5 + 7\sqrt{5}) = x^2 - 10x - 220 = 0.$

16. $(x - a - b)(x - a + b) = x^2 - 2ax + a^2 - b^2 = 0.$

17. $(x - a - \sqrt{a^2 - b^2})(x - a + \sqrt{a^2 - b^2}) =$
$$x^2 - 2ax + b^2 = 0.$$

§ 202.

1. Clear of fractions, $x - a$, L.C.M.;
$$\sqrt{x} - \sqrt{a} + \sqrt{x} + \sqrt{a} = 2\sqrt{a} - 2\sqrt{x};$$
transpose and eliminate, $4\sqrt{x} = 2\sqrt{a}$,
$$2\sqrt{x} = \sqrt{a};$$
square, $4x = a,\ x = \frac{a}{4}.$

2. Clear of fractions, $a\sqrt{x^2 + a} = x\sqrt{a^2 - x}$;
square, $a^2x^2 + a^3 = a^2x^2 - x^3$,
$$x^3 = -a^3,$$
$$x = -a.$$

3. Change sign of radical and multiply,
$$\sqrt{x+3} + \sqrt{x-4} - 1 = 0,$$
$$\sqrt{x+3} - \sqrt{x-4} - 1 = 0;$$
transpose, $(\sqrt{x+3} - 1)^2 - x + 4 = 0,$
$$x + 3 - 2\sqrt{x+3} + 1 - x + 4 = 0,$$
$$-2\sqrt{x+3} + 8 = 0,$$
$$2\sqrt{x+3} + 8 = 0,$$
$$-4(x+3) + 64 = 0,$$
$$4x + 12 = 64,$$
$$4x = 52,$$
$$x = 13.$$

4. Transpose, $\sqrt{x+14} - 14 + \sqrt{x-14} = 0$;
change sign of $\sqrt{x-14}$ and multiply,
$(\sqrt{x+14} - 14)^2 - (x - 14) = 0,$
or $x + 14 - 28\sqrt{x+14} + 196 - x + 14 = 0;$
eliminate, $224 - 28\sqrt{x+14} = 0;$
$\div 28$, $8 - \sqrt{x+14} = 0;$
change sign of radical and multiply,
$$64 - x - 14 = 0,$$
$$x = 50.$$

5. Transpose, $(3 - x)^{\frac{1}{2}} = (3 + x^2)^{\frac{1}{4}};$
square, $3 - x = (3 + x^2)^{\frac{1}{2}};$
square, $9 - 6x + x^2 = 3 + x^2;$
transpose, $6 = 6x,\ x = 1.$

6. Transpose, $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}-2\sqrt{x+\frac{1}{2}a}=0$;
change sign and multiply,

$$\left(\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}\right)^2-4\left(x+\tfrac{1}{2}a\right)=0,$$

or $a+\sqrt{x}+2\sqrt{a^2-x}+a-\sqrt{x}-4x-2a=0$;
transpose and eliminate, $-4x+2\sqrt{a^2-x}=0$;
$\div 2$, $\quad -2x+\sqrt{a^2-x}=0$;
change sign and multiply, $4x^2-(a^2-x)=0$,
or $4x^2+x=a^2$;
complete square, $4x^2+x+\dfrac{1}{16}=a^2+\dfrac{1}{16}=\dfrac{16a^2+1}{16}$;
extract $\sqrt{\ }$, $\quad 2x+\dfrac{1}{4}=\dfrac{\sqrt{16a^2+1}}{4}$,

$$x=\tfrac{1}{8}\left(-1\pm\sqrt{16a^2+1}\right).$$

7. Clear of fractions, $x-4$, L.C.M;

$$\sqrt{x}-2+\sqrt{x}-\sqrt{x}-2=0,$$
$$\sqrt{x}=4,\ x=16.$$

8. Clear of fractions, $10x-18-2\sqrt{5x}-6=5x-9$;
transpose, $\quad 5x-15-2\sqrt{5x}=0$;
change sign of radical and multiply,

$$(5x-15)^2-20x=0,$$
$$25x^2-150x+225-20x=0,$$
$$25x^2-170x=-225;$$

$\div 25$, $\quad x^2-\frac{34}{5}x=-9$;
complete, $x^2-\frac{34}{5}x+\frac{289}{25}=\frac{289}{25}-9=\frac{64}{25}$;
extract $\sqrt{\ }$, $\quad x-\frac{17}{5}=\pm\frac{8}{5}$,

$$x=5\text{ or }\tfrac{9}{5}.$$

9. Clear of fractions, $a^2-2x+x=b\sqrt{a^2-2x}$;
transpose and eliminate, $a^2-x-b\sqrt{a^2-2x}=0$;
change sign and multiply, $(a^2-x)^2-b^2(a^2-2x)=0$,
or $a^4-2a^2x+x^2-a^2b^2+2b^2x=0$;
transpose, $\quad x^2-2a^2x+2b^2x=a^2b^2-a^4$,

$$x^2-2(a^2-b^2)4x=a^2b^2-a^4;$$

complete,
$x^2-2(a^2-b^2)x+(a^2-b^2)^2=a^4-a^2b^2+b^4-a^4$;
extract $\sqrt{\ }$, $x-(a^2-b^2)=\sqrt{b^4-a^2b^2}=\pm b\sqrt{b^2-a^2}$,

$$x=a^2-b^2\pm b\sqrt{b^2-a^2}.$$

10. Rationalize the numerator, $\dfrac{x^2 - x}{(x - \sqrt{x})^2} = \dfrac{x^2 - x}{4}$;

$\div\, x^2 - x$, $\dfrac{1}{(x - \sqrt{x})^2} = \dfrac{1}{4}$;

extract $\sqrt{}$, $\dfrac{1}{x - \sqrt{x}} = \dfrac{1}{2}$,

$x - 2 - \sqrt{x} = 0$;

change sign of radical and multiply, $(x - 2)^2 - x$

or $x^2 - 4x + 4 - x = 0$;

transpose, $x^2 - 5x = -4$;

complete, $x^2 - 5x + \frac{25}{4} = \frac{25}{4} - 4 = \frac{9}{4}$;

extract $\sqrt{}$, $x - \frac{5}{2} = \pm \frac{3}{2}$,

$x = 4$ or 1.

11. Clear of fr., $\sqrt{x + ax - 1 - a} = \sqrt{x - a} + \sqrt{ax - 1}$;

transpose, $\sqrt{x + ax - 1 - a} - \sqrt{x - a} - \sqrt{ax - 1} = 0$;

change sign and multiply,

$x + ax - 1 - a - [x - a + ax - 1 + 2\sqrt{(x - a)(ax - 1)}] = 0$;

cancelling equal terms,

$2\sqrt{(x - a)(ax - 1)} = 0,\ (x - a)(ax - 1) = 0$;

$\therefore x = a$ or $x = \dfrac{1}{a}$.

§ 203.

1. From (2), $y = 12 - 2x$,

$y^2 = 144 - 48x + 4x^2$;

sub. in (1), $x^2 - 2x(12 - 2x) + 576 - 192x + 16x^2 = 21$,

or $x^2 - 24x + 4x^2 + 16x^2 - 192x = 21 - 576$,

$21x^2 - 216x = -555$;

$\div\, 21$, $x^2 - \frac{72}{7}x = -\frac{185}{7}$

complete, $x^2 - \frac{72}{7}x + (\frac{36}{7})^2 = -\frac{185}{7} + \frac{1296}{49} = \frac{1}{49}$;

extract $\sqrt{}$, $x - \frac{36}{7} = \pm \frac{1}{7}$,

$x = \frac{36}{7} \pm \frac{1}{7} = \frac{37}{7}$ or 5;

$y = 12 - 2x = 12 - \frac{74}{7}$ or $-10 = \frac{10}{7}$ or 2.

2. From (2), $x = -4 - y$,

$x^2 = 16 + 8y + y^2$;

sub. in (1),

$3(16 + 8y + y^2) - 2y^2 + 5(-4 - y) - 2y = 28$,

$48 + 24y + 3y^2 - 2y^2 - 20 - 5y - 2y = 28$;

transpose, $y^2 + 17y = 0$;

complete, $\quad y^2 + 17y + \frac{289}{4} = \frac{289}{4},$

$$y + \tfrac{17}{2} = \pm \tfrac{17}{2},$$

$$y = -\tfrac{17}{2} \pm \tfrac{17}{2} = 0 \text{ or } -17;$$

$$x = -4 - y = -4 - 0 \text{ or } +17 = -4 \text{ or } 13.$$

3. From (2), $\quad x = -2y;$

sub. in (1), $\quad -10y^2 + 7y^2 + 2y - y = 72,$

$$\text{or } -3y^2 + y = 72;$$

$\div -3,$ $\quad y^2 - \dfrac{y}{3} = -24,$

$$y^2 - \frac{y}{3} + \frac{1}{36} = -24 + \frac{1}{36} = -\frac{863}{36};$$

extract $\sqrt{\ }$, $\quad y - \frac{1}{6} = \pm \sqrt{-\frac{863}{36}} = \pm \frac{1}{6}\sqrt{-863},$

$$y = \tfrac{1}{6} \pm \tfrac{1}{6}\sqrt{-863} = \tfrac{1}{6}(1 \pm \sqrt{-863});$$

$$x = -2y = -\tfrac{1}{3} \mp \tfrac{1}{3}\sqrt{-863} = -\tfrac{1}{3}(1 \pm \sqrt{-863}).$$

4. Transpose (2), $\quad 7x = 17 + 4y,$

$$x = \frac{17 + 4y}{7};$$

sub. in (1), $\quad 3\left(\dfrac{17 + 4y}{7}\right)^2 + 2y^2 = 813,$

$$3\left(\frac{289 + 136y + 16y^2}{49}\right) + 2y^2 = 813;$$

clear of fr., $\quad 867 + 408y + 48y^2 + 98y^2 = 39\,837;$

transpose, $\quad 146y^2 + 408y = 38\,970;$

$\div 146,$ $\quad y^2 + \frac{204}{73}y = \frac{19\,485}{73};$

complete, $\quad y^2 + \frac{204}{73}y + \left(\frac{102}{73}\right)^2 = \frac{19\,485}{73} + \frac{10\,404}{5329}$

$$= \frac{1\,432\,809}{5329};$$

extract $\sqrt{\ }$, $\quad y + \frac{102}{73} = \pm \frac{1197}{73},$

$$y = -\tfrac{102}{73} \pm \tfrac{1197}{73} = 15 \text{ or } -17\tfrac{58}{73};$$

$$x = \frac{17 + 4y}{7} = \frac{17 + 60}{7} = \frac{77}{7} = 11,$$

$$\text{or } x = \frac{17 - \frac{5196}{73}}{7} = \frac{1241 - 5196}{7.73} = -\frac{3955}{7.73} = -\frac{565}{73}$$

$$= -7\tfrac{54}{73}. \quad \text{Ans.}$$

5. Clear of fractions, $\quad 12x^2 - 12y^2 = 7xy \quad (3);$

from (1), $\quad x = 7 - y;$

sub. in (3), $\quad 12(49 - 14y + y^2) - 12y^2 = 49y - 7y^2,$

$$\text{or } 588 - 168y + 12y^2 - 12y^2 = 49y - 7y^2;$$

transpose and eliminate, $\quad 7y^2 - 217y = -588;$

$\div 7,$ $\quad y^2 - 31y = -84;$

complete, $\quad y^2 - 31y + \frac{961}{4} = \frac{961}{4} - 84 = \frac{625}{4};$

extract $\sqrt{\ }$, $\quad y - \frac{31}{2} = \pm \frac{25}{2}. \quad y = 28 \text{ or } 3;$

$$x = 7 - y = -21 \text{ or } 4.$$

§ 204.

1. $\times$ (2) by 6, $\quad 6x^2 + 12x - 18y = 108 \ (3);$

(3) − (1), $\quad 15x - 14y = 83,$

$$15x = 83 + 14y \ (4),$$

$$x = \frac{83 + 14y}{15}.$$

Sub. in (2), $\left(\frac{83 + 14y}{15}\right)^2 + 2\left(\frac{83 + 14y}{15}\right) - 3y = 18,$

$$\frac{6889 + 2324y + 196y^2}{225} + \frac{166 + 28y}{15} - 3y = 18;$$

clear of fractions, $6889 + 2324y + 196y^2 + 2490 + 420y - 675y = 4050;$

transpose, $\quad 196y^2 + 2069y = -5329$

$$y^2 + \tfrac{2069}{196}y = -\tfrac{5329}{196};$$

complete, $\quad y^2 + \tfrac{2069}{196}y + \left(\tfrac{2069}{392}\right)^2 = -\tfrac{5329}{196} + \tfrac{4\,280\,761}{153\,664}$

$$= \tfrac{102\,825}{153\,664};$$

extract $\surd$, $\quad y + \tfrac{2069}{392} = \pm .818,$

$$y = -5.278 \pm .818 = -4.46 \text{ or } -6.096.$$

Sub. in (4), $\quad 15x = 83 - 62.44,$

$$15x = 20.56,$$

$$x = 1.37,$$

$$\text{or } 15x = 83 - 85.344 = -2.344,$$

$$x = -.156.$$

2. $\times$ (2) by 2, $\quad 2y^2 + 6x - 8y = 36 \ (3);$

(3) − (1), $\quad 6x - 9y = 8,$

$$6x = 8 + 9y,$$

$$x = \frac{8 + 9y}{6} \ (4).$$

Sub. in (2), $\quad y^2 + 3\left(\frac{8 + 9y}{6}\right) - 4y = 18,$

$$y^2 + \frac{8 + 9y}{2} - 4y = 18;$$

clear of fractions, $2y^2 + 8 + 9y - 8y = 36;$

transpose and ÷ 2, $\quad y^2 + \frac{y}{2} = 14;$

complete, $\quad y^2 + \frac{y}{2} + \frac{1}{16} = 14 + \frac{1}{16} = \frac{225}{16}.$

extract $\surd$, $\quad y + \tfrac{1}{4} = \pm \tfrac{15}{4},$

$$y = -\tfrac{1}{4} \pm \tfrac{15}{4} = 3\tfrac{1}{2} \text{ or } -4.$$

Sub. in (4), $\quad x = \frac{8 + \frac{63}{2}}{6} \text{ or } \frac{8 - 36}{6},$

$$x = 6\tfrac{7}{12} \text{ or } -4\tfrac{2}{3}.$$

3. $\times$ (1) by 3, $\quad 3xy + 18x + 21y = 198$ (3);

$$
\begin{aligned}
(3) - (1), \quad && 16x + 16y &= 128; \\
\div 16, \quad && x + y &= 8, \\
&& x &= 8 - y; \\
\text{sub. in (1),} \quad && 8y - y^2 + 48 - 6y + 7y &= 66, \\
&& y^2 - 9y &= -18; \\
\text{complete,} \quad && y^2 - 9y + \tfrac{81}{4} = \tfrac{81}{4} - 18 &= \tfrac{9}{4}; \\
\text{extract } \surd, \quad && y - \tfrac{9}{2} &= \pm \tfrac{3}{2}, \\
&& y &= 6 \text{ or } 3; \\
&& x = 8 - y &= 2 \text{ or } 5.
\end{aligned}
$$

§ 205.

1. $\times$ (1) by 4, $\quad 4x^2 - 4xy + 4y^2 - 12 = 0$ (3);

$\times$ (2) by 3, $\quad 3x^2 - 6xy + 12y^2 - 12 = 0$ (4);

$$
\begin{aligned}
(3) - (4), \quad && x^2 + 2xy - 8y^2 &= 0, \\
&& x^2 + 2xy &= 8y^2; \\
\text{complete,} \quad && x^2 + 2xy + y^2 &= 9y^2; \\
\text{extract } \surd, \quad && x + y &= \pm 3y, \\
&& x = -y \pm 3y &= 2y \text{ or } -4y; \\
\text{sub. in (1), } x = 2y, \quad && 4y^2 - 2y^2 + y^2 &= 3, \\
&& 3y^2 &= 3, \\
&& y^2 &= 1, \\
&& y &= 1; \\
\text{sub. in (1), } x = -4y, \quad && 16y^2 + 4y^2 + y^2 &= 3,
\end{aligned}
$$

$$21y^2 = 3, \quad y^2 = \tfrac{1}{7}, \quad y = \pm \sqrt{\tfrac{1}{7}};$$

$$x = 2y = 2 \text{ or } 2\sqrt{\tfrac{1}{7}},$$

$$x = -4y = -4 \text{ or } -4\sqrt{\tfrac{1}{7}}.$$

2. $\times$ (2) by 2, $\quad 2x^2 + 6xy - 8y^2 + 2 = 0$ (3);

$$
\begin{aligned}
(1) + (3), \quad && 4x^2 + 9xy - 9y^2 &= 0, \\
&& 4x^2 + 9xy &= 9y^2;
\end{aligned}
$$

complete, $4x^2 + 9xy + \tfrac{81}{16}y^2 = 9y^2 + \tfrac{81}{16}y^2 = \tfrac{225}{16}y^2$;

extract $\surd$, $\quad 2x + \tfrac{9}{4}y = \pm \tfrac{15}{4}y$,

$$2x = -\tfrac{9}{4}y \pm \tfrac{15}{4}y = \tfrac{3}{2}y \text{ or } -6y,$$

$$x = \tfrac{3}{4}y \text{ or } -3y.$$

Sub. 1st value in (2), $\tfrac{9}{16}y^2 + \tfrac{9}{4}y^2 - 4y^2 + 1 = 0$;

transpose, $-\tfrac{19}{16}y^2 = -1$,

$$y^2 = +\tfrac{16}{19},$$

$$y = \pm\sqrt{\tfrac{16}{19}} = \sqrt{\frac{16 \times 19}{19 \times 19}} = \pm \tfrac{4}{19}\sqrt{19}.$$

Sub. 2d value in (2), $\quad 9y^2 - 9y^2 - 4y^2 + 1 = 0$;

transpose, $\quad -4y^2 = -1;$

$\div -4, \quad y^2 = \frac{1}{4},$

$y = \pm \frac{1}{2};$

$x = \frac{3}{4}y = \frac{3}{4}(\pm \frac{4}{19}\sqrt{19}) = \pm \frac{3}{19}\sqrt{19},$

$x = -3y = -3(\pm \frac{1}{2}) = \mp \frac{3}{2}.$

§ 207.

1. From (1), $\quad y(x+y) = 14 \quad (3);$
(1) + (2), $\quad x^2 + 2xy + y^2 = 49;$
extract $\sqrt{\ }$, $\quad x + y = \pm 7 \quad (4);$
sub. in (3), $\quad y(\pm 7) = 14,$
$\pm 7y = 14;$
$y = \pm 2;$
sub. in (4), $\quad x \pm 2 = \pm 7,$
$x = \pm 5.$

2. (1) − (2), $\quad 4x^2 - 4xy + y^2 = 169;$
extract $\sqrt{\ }$, $\quad 2x - y = \pm 13 \quad (3);$
factor (1), $\quad 2x(2x - y) = 208 \quad (4);$
sub. (3) in (4), $\quad 2x(\pm 13) = 208,$
$\pm 26x = 208,$
$x = \pm 8;$
sub. in (3), $\quad \pm 16 - y = \pm 13;$
transpose and change signs, $y = \pm 16 \mp 13 = \pm 3.$

3. (1) − (2), $\quad x^2 + y - y^2 - x = 4x - 4y;$
transpose, $\quad x^2 - y^2 = 5x - 5y;$
$\div x - y$, $\quad x + y = 5,$
$x = 5 - y;$
sub. in (1), $\quad (5-y)^2 + y = 4(5-y),$
$25 - 10y + y^2 + y = 20 - 4y;$
transpose, $\quad y^2 - 5y = -5;$
complete, $\quad y^2 - 5y + \frac{25}{4} = -5 + \frac{25}{4} = \frac{5}{4};$
extract $\sqrt{\ }$, $\quad y - \frac{5}{2} = \pm\sqrt{\frac{5}{4}} = \pm \frac{1}{2}\sqrt{5},$
$y = \frac{5}{2} \pm \frac{1}{2}\sqrt{5} = \frac{1}{2}(5 \pm \sqrt{5});$
sub. in (3), $\quad x = 5 - \frac{5}{2} \mp \frac{1}{2}\sqrt{5},$
$x = \frac{5}{2} \mp \frac{1}{2}\sqrt{5} = \frac{1}{2}(5 \mp \sqrt{5}).$

4. (1) + (2), $\quad 2x^3 + 2y^3 = 702$, or $x^3 + y^3 = 351 \quad (3);$
(1) − (2), $\quad 6x + 6y = 54$, or $x + y = 9; \therefore x = 9 - y;$
sub. in (3), $\quad 729 - 243y + 27y^2 - y^3 + y^3 = 351,$
or $27y^2 - 243y = -378;$
$\div 27$, $\quad y^2 - 9y = -14;$

complete, $y^2 - 9y + \frac{81}{4} = \frac{81}{4} - 14 = \frac{25}{4}$;
extract $\sqrt{\ }$, $y - \frac{9}{2} = \pm \frac{5}{2}$,
$y = 7$ or 2;
$x = 9 - y = 2$ or 7.

5. Square (2), $x^2 + 2xy + y^2 = 144$ (3);
(3) − (1), $2xy = 70$,
$xy = 35$,
$x = \dfrac{35}{y}$;
sub. in (2), $\dfrac{35}{y} + y = 12$;
clear of fractions, $35 + y^2 = 12y$;
transpose, $y^2 - 12y = -35$;
complete, $y^2 - 12y + 36 = -35 + 36 = 1$;
extract $\sqrt{\ }$, $y - 6 = \pm 1$,
$y = 6 \pm 1 = 7$ or 5;
sub. in (2), $x + 7 = 12$,
or $x + 5 = 12$;
$\therefore x = 5$ or 7.

6. (1) − (2), $xy + y^2 = -14$ (3);
(1) + (3), $x^2 + 2xy + y^2 = 49$;
extract $\sqrt{\ }$, $x + y = \pm 7$ (4);
factor (1), $x(x + y) = 63$ (5);
sub. (4) in (5), $x(\pm 7) = 63$,
$x = \pm 9$;
sub. in (4), $\pm 9 + y = \pm 7$;
transpose, $y = \pm 7 \mp 9 = \mp 2$.

7. Clear of fractions, $\sqrt{x} + \sqrt{y} = 4\sqrt{x} - 4\sqrt{y}$;
transpose, $5\sqrt{y} = 3\sqrt{x}$;
square, $25y = 9x$,
$y = \dfrac{9x}{25}$.
sub. in (2), $x^2 - \dfrac{81x^2}{625} = 544$;
clear of fractions, $625x^2 - 81x^2 = 340\,000$,
$544x^2 = 340\,000$,
$x^2 = 625$,
$x = \pm 25$;
$y = \dfrac{9x}{25} = \pm 9$.

8. (1) + (2), $x^2 + 2xy + y^2 = a + b$;
extract $\sqrt{\ }$, $x + y = \pm\sqrt{a+b}$,
$$x = \pm\sqrt{a+b} - y;$$
sub. in (2), $y^2 + y(\pm\sqrt{a+b} - y) = b$,
$$y^2 \pm y\sqrt{a+b} - y^2 = b,$$
$$\pm y\sqrt{a+b} = b,$$
$$y = \pm\frac{b}{\sqrt{a+b}};$$
$$x = \pm\sqrt{a+b} - y = \pm\sqrt{a+b} \mp \frac{b}{\sqrt{a+b}},$$
$$x = \frac{\pm(a+b) \mp b}{\sqrt{a+b}} = \frac{\pm a \pm b \mp b}{\sqrt{a+b}} = \pm\frac{a}{\sqrt{a+b}}.$$

9. Let $x = uy$;
then (1) becomes $u^2y^2 + uy^2 = 10$,
$$y^2 = \frac{10}{u^2+u} \quad (3);$$
(2) becomes $y^2 + u^2y^2 = 5$,
$$y^2 = \frac{5}{1+u^2}.$$
Now we have two values of y^2 which must be equal,
that is, $\frac{10}{u^2+u} = \frac{5}{u^2+1}$;
$\times (u^2+1)$, $\frac{10}{u} = 5$,
$$10 = 5u,$$
$$u = 2,$$
$$y^2 = \tfrac{10}{10} = 1,\ y = 1;$$
sub. in (2), $1 + x^2 = 5,\ x^2 = 4,\ x = 2$.

10. Square, $x^2 = a^2x + a^2y$ (3)
and $y^2 = b^2x + b^2y$ (4);
(3) − (4) $= x^2 - y^2 = a^2x - b^2x + a^2y - b^2y$,
or $x^2 - y^2 = x(a^2 - b^2) + y(a^2 - b^2)$,
or $x^2 - y^2 = (a^2 - b^2)(x + y)$;
$\div\ x + y$, $x - y = a^2 - b^2$,
$$x = a^2 - b^2 + y;$$
sub. in (4), $y^2 = b^2(a^2 - b^2 + y) + b^2y$,
or $y^2 = a^2b^2 - b^4 + b^2y + b^2y$;
transpose, $y^2 - 2b^2y = a^2b^2 - b^4$;
complete, $y^2 - 2b^2y + b^4 = a^2b^2 - b^4 + b^4 = a^2b^2$;

extract $\sqrt{\ }$, $\quad y - b^2 = \pm ab,$

$$y = b^2 \pm ab;$$

$$x = a^2 - b^2 + y = a^2 - b^2 + b^2 \pm ab = a^2 \pm ab.$$

11. Square, $x^2(x+y) = 144,\ x^3 + x^2y = 144\ (3);$

$$y^2(x+y) = 225,\ xy^2 + y^3 = 225\ (4).$$

Let $\quad x = uy;$

then (3) becomes $u^3y^3 + u^2y^3 = 144,$

$$y^3 = \frac{144}{u^3+u^2};$$

and (4) becomes $\quad uy^3 + y^3 = 225,$

$$y^3 = \frac{225}{u+1}\ (5);$$

by axiom, $\quad \dfrac{144}{u^3+u^2} = \dfrac{225}{u+1};$

multiply by $u + 1 = \dfrac{144}{u^2} = 225$, or $225\,u^2 = 144,$

or $\quad 15\,u = 12,\ u = \frac{12}{15}$ or $\frac{4}{5};$

sub. in (5), $\quad y^3 = \dfrac{225}{\frac{4}{5}+1} = \dfrac{225}{\frac{9}{5}} = 125,$

hence $\quad y = 5;$

sub. in (4), $\quad 25\,x + 125 = 225,$

$$25\,x = 100,$$

$$x = 4.$$

12. $\div$ (1) by 2, $\quad x^2 + y^2 = \dfrac{x+y}{2};$

also, $\quad x^2 + y^2 = x - y;$

$$\therefore \frac{x+y}{2} = x - y;$$

clear of fractions, $\quad x + y = 2\,x - 2\,y;$

transpose, $\quad x = 3\,y;$

sub. in (2), $\quad 9\,y^2 + y^2 = 3\,y - y,$

$$\text{or } 10\,y^2 = 2\,y;$$

$\div\ y$, $\quad 10\,y = 2,$

$$y = \tfrac{2}{10} \text{ or } \tfrac{1}{5};$$

$$x = 3\,y = \tfrac{3}{5}.$$

13. $\div$ (1) by 5, $\quad x^2 - y^2 = \dfrac{x+y}{5};$

$\div$ (2) by 3, $\quad x^2 - y^2 = \dfrac{x-y}{3};$

$$\therefore \frac{x+y}{5} = \frac{x-y}{3};$$

clear of fractions, $3x + 3y = 5x - 5y$;
transpose, $8y = 2x$,
$x = 4y$;
sub. in (2), $48y^2 - 3y^2 = 4y - y$,
$45y^2 = 3y$,
$45y = 3$,
$y = \frac{3}{45}$ or $\frac{1}{15}$;
$x = 4y = \frac{4}{15}$.

14. $\times$ (2) by 2, $2xy + 2yz + 2zx = 34$ (4);
(4) + (1) $= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 64$ (5);
extract $\sqrt{\ }$, $x + y + z = 8$ (6);
(6) + (3), $2x = 10$, $x = 5$;
sub. in (1), $y^2 + z^2 = 5$ (7);
sub. in (3), $-y - z = -3$, or $y + z = 3$, $y = 3 - z$;
sub. this value of y in (7), $9 - 6z + z^2 + z^2 = 5$,
$2z^2 - 6z = -4$,
$z^2 - 3z = -2$;
complete, $z^2 - 3z + \frac{9}{4} = \frac{9}{4} - \frac{8}{4} = \frac{1}{4}$;
extract $\sqrt{\ }$, $z - \frac{3}{2} = \pm \frac{1}{2}$,
$z = 2$ or 1;
$y = 3 - z = 1$ or 2.

15. Square (1), $\dfrac{6y}{x-y} - 6 + 9\dfrac{x-y}{6y} = 4$;

$\times \dfrac{x-y}{6y}$, $9\left(\dfrac{x-y}{6y}\right)^2 - 10\dfrac{x-y}{6y} = -1$.

Solve as a quadratic in which $\dfrac{x-y}{6y}$ is the unknown quantity.

For brevity put $z = \dfrac{x-y}{6y}$;
then $9z^2 - 10z = -1$;
completing sq., $9z^2 - 10z + \frac{100}{36} = \frac{100}{36} - 1 = \frac{64}{36}$;
extract $\sqrt{\ }$, $3z - \frac{10}{6} = \pm \frac{8}{6} = \pm \frac{4}{3}$;

$$\therefore z = \frac{5 \pm 4}{9} = 1, \text{ or } \tfrac{1}{9} = \frac{x-y}{6y},$$

$$x - y = 6y \text{ or } \frac{6y}{9},$$

$$x = 7y \text{ or } \tfrac{5}{3}y; \qquad (a)$$

$\times$ (2) by $x - y$, $x^2 - y^2 - 2\sqrt{x^2 - y^2} = 8$.
Solving this as a quadratic for $x^2 - y^2$, we get
$x^2 - y^2 = 16 = (x + y)(x - y)$.

From (a), if $x = 7y$, $(x+y)(x-y) = 8y \times 6y = 48y^2$;

$$\therefore 48y^2 = 16,\ y = \sqrt{\tfrac{1}{3}},\ x = \frac{7}{\sqrt{3}}.$$

If $x = \frac{5}{3}y$, then $(x+y)(x-y) = \frac{8}{3}y \times \frac{2}{3}y = \frac{16}{9}y^2$;

$$\therefore \tfrac{16}{9}y^2 = 16,\ y^2 = 9,\ y = 3,\ x = 5.$$

The values corresponding to $x = 7y$ are only admissible when the radicals in the first equation are taken negatively.

16.

$$\begin{aligned} \text{Let } r &= \text{rate}; \\ t &= \text{time}; \\ \$7100 - \$5000 &= \$2100 \text{ interest}; \end{aligned}$$

then (I) $\$5000 \times \dfrac{r}{100} \times t = \$2100,$

$$\begin{aligned} 50rt &= \$2100, \\ rt &= 42,\ r = \frac{42}{t}; \end{aligned}$$

(II) $\$7800 - \$5000 = \$2800$ interest,

and $\$5000 \times \dfrac{r+1}{100}(t+1) = \$2800$;

$$\begin{aligned} 50(rt + r + t + 1) &= \$2800, \\ rt + r + t + 1 &= 56, \\ rt + r + t &= 55, \end{aligned}$$

sub. $rt = 42$, $\quad t + r = 13\ (1)$;

sub. value of r in (1), $t + \dfrac{42}{t} = 13$;

clear of fractions, $t^2 + 42 = 13t$, or $t^2 - 13t = -42$;

complete, $t^2 - 13t + (\frac{13}{2})^2 = \frac{169}{4} - 42 = \frac{1}{4}$;

extract $\sqrt{\ }$, $\quad t - \frac{13}{2} = \pm\frac{1}{2}$, $t = 7$ or 6;

$$r = \frac{42}{t} = 6 \text{ or } 7.$$

17. Let

$$\begin{aligned} x &= \text{time of first}; \\ y &= \text{rate}; \\ x - 5 &= \text{time of second}; \\ y + 3 &= \text{rate}; \\ x - 7 &= \text{time of third}. \end{aligned}$$

$xy = (x-5)(y+3)$, since the distance travelled was the same, and

$xy = 10(x-7)$, $xy = 10x - 70$, or $xy - 10x = -70$,

$$x = \frac{-70}{y-10};$$

also, $\quad xy = xy - 5y + 3x - 15,$

or $\quad 3x = 15 + 5y,\ x = \dfrac{15 + 5y}{3};$

making these values of x equal,

$$\frac{15 + 5y}{3} = \frac{-70}{y - 10};$$

clear of fractions, $15y - 150 + 5y^2 - 50y = -210,$

or $\quad 5y^2 - 35y = -60,\ y^2 - 7y = -12;$

complete, $\quad y^2 - 7y + \frac{49}{4} = \frac{49}{4} - 12 = \frac{1}{4};$

extract $\sqrt{\ }$, $y - \frac{7}{2} = \pm\frac{1}{2},\ y = 4$ or $3;$

$$x = \frac{-70}{y - 10} = \frac{-70}{-7 \text{ or } -6} = 10 \text{ or } 11\tfrac{2}{3}.$$

Distance, 30 or $46\frac{2}{3}$ miles.

18. $\quad$ Let x = base;

y = altitude;

then $\quad x^2 + y^2 = a^2$ (1),

and $\quad \dfrac{xy}{2} = b^2$, or $xy = 2b^2$ (2);

$\times$ (2) by 2, $\quad 2xy = 4b^2$ (3);

(1) + (3), $\quad x^2 + 2xy + y^2 = a^2 + 4b^2;$

extract $\sqrt{\ }$, $\quad x + y = \sqrt{a^2 + 4b^2}$ (4);

(1) − (3), $\quad x^2 - 2xy + y^2 = a^2 - 4b^2;$

extract $\sqrt{\ }$, $\quad x - y = \sqrt{a^2 - 4b^2}$ (5);

(4) + (5), $\quad 2x = \sqrt{a^2 + 4b^2} + \sqrt{a^2 - 4b^2},$

$$x = \frac{\sqrt{a^2 + 4b^2} + \sqrt{a^2 - 4b^2}}{2};$$

(4) − (5), we get $\quad y = \dfrac{\sqrt{a^2 + 4b^2} - \sqrt{a^2 - 4b^2}}{2}.$

19. $\quad$ Let x = 1st number,

y = 2d number;

then $\quad xy = x + y$ (1),

$x + y = x^2 - y^2$ (2);

$\div$(2) by $x + y$, $\quad 1 = x - y$, or $x = 1 + y$ (3);

sub. in (1), $\quad y(1 + y) = 1 + y + y,$

$y + y^2 = 1 + 2y;$

transpose, $\quad y^2 - y = 1;$

complete sq., $\quad y^2 - y + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4};$

extract $\sqrt{\ }$, $\quad y - \frac{1}{2} = \sqrt{\frac{5}{4}} = \pm\frac{1}{2}\sqrt{5},$

$y = \frac{1}{2} \pm \frac{1}{2}\sqrt{5} = \frac{1}{2}(1 \pm \sqrt{5});$

sub. in (3), $\quad x = 1 + \frac{1}{2} \pm \frac{1}{2}\sqrt{5},$

$x = \frac{3}{2} \pm \frac{1}{2}\sqrt{5} = \frac{1}{2}(3 \pm \sqrt{5}).$

20.

$$\begin{aligned}
&& \text{Let } x &= \text{the greater,} \\
&& y &= \text{the less;} \\
&\text{then} & xy &= 216 \;\; (1), \\
&\text{and} & (x-4)(y+3) &= 240, \\
&\text{or} & xy+3x-4y-12 &= 240 \;\; (2); \\
&\text{from (1),} & x &= \frac{216}{y} \;\; (3); \\
&\text{sub. in (2),} & 216+\frac{648}{y}-4y-12 &= 240; \\
&\text{transpose,} & \frac{648}{y}-4y &= 36; \\
&\text{clear of fr.,} & 648-4y^2 &= 36y; \\
&\text{transpose,} & 4y^2+36y &= 648; \\
&\div 4, & y^2+9y &= 162; \\
&\text{complete sq.,} & y^2+9y+\tfrac{81}{4} &= 162+\tfrac{81}{4} = \tfrac{729}{4}; \\
&\text{extract } \sqrt{\ }, & y+\tfrac{9}{2} &= \pm\tfrac{27}{2}, \\
&& y &= 9 \text{ or } -18; \\
&\text{sub. in (3),} & x &= \frac{216}{9} \text{ or } \frac{216}{-18} = 24 \text{ or } -12.
\end{aligned}$$

21.

$$\begin{aligned}
&& \text{Let } x &= \text{first,} \\
&& y &= \text{second;} \\
&\text{then} & x+y &= 74 \;\; (1), \\
&& \sqrt{x}+\sqrt{y} &= 12 \;\; (2); \\
&\text{transpose,} & \sqrt{x} &= 12-\sqrt{y}; \\
&\text{square,} & x &= 144-24\sqrt{y}+y; \\
&\text{transpose,} & x-y &= 144-24\sqrt{y} \;\; (3); \\
&(1)-(3), & 2y &= 74-144+24\sqrt{y} = -70+24\sqrt{y}, \\
&& y &= -35+12\sqrt{y}; \\
&\text{transpose,} & y-12\sqrt{y} &= -35; \\
&\text{comp. sq.,} & y-12\sqrt{y}+36 &= -35+36 = 1; \\
&\text{extract } \sqrt{\ }, & \sqrt{y}-6 &= 1, \\
&& \sqrt{y} &= 7 \text{ or } 5, \\
&& y &= 49 \text{ or } 25; \\
&\text{sub. in (1),} & x+49 \text{ or } 25 &= 74, \\
&& x &= 25 \text{ or } 49.
\end{aligned}$$

22.

$$\begin{aligned}
&& \text{Let } x &= \text{first,} \\
&& y &= \text{second;} \\
&\text{then} & x+y &= 72, \\
&\text{and} & x^{\frac{1}{3}}+y^{\frac{1}{3}} &= 6; \\
&\text{transpose,} & x^{\frac{1}{3}} &= 6-y^{\frac{1}{3}}; \\
&\text{cube,} & x &= 216-108y^{\frac{1}{3}}+18y^{\frac{2}{3}}-y;
\end{aligned}$$

sub. in (1), $216 - 108y^{\frac{1}{3}} + 18y^{\frac{2}{3}} - y + y = 72$;
transpose, $18y^{\frac{2}{3}} - 108y^{\frac{1}{3}} = -144$;
$\div 18$, $\quad y^{\frac{2}{3}} - 6y^{\frac{1}{3}} = -8$;
complete sq., $\quad y^{\frac{2}{3}} - 6y^{\frac{1}{3}} + 9 = -8 + 9 = 1$;
extract $\sqrt{\ }$, $\quad y^{\frac{1}{3}} - 3 = \pm 1$,
$$y^{\frac{1}{3}} = 4 \text{ or } 2,$$
$$y = 64 \text{ or } 8;$$
sub. in (1), $x + 64 \text{ or } 8 = 72$,
$$x = 8 \text{ or } 64.$$

23. Let $x =$ the length of the rectangle of double area;
$y =$ " breadth " " " " "
mn is the area of the first rectangle;
$2mn$ " " " second " or xy;
$2(m+n)$ " perimeter of the first rectangle;
$4(m+n)$ " " " second " or $2(x+y)$.
Then $\quad xy = 2mn \quad (1)$,
$$x + y = 2(m+n) \quad (2);$$
squaring (2), $x^2 + 2xy + y^2 = 4m^2 + 8mn + 4n^2$;
(1) $\times 4$, $\quad 4xy = 8mn$;
subtract, $\quad x^2 - 2xy + y^2 = (x-y)^2 = 4(m^2 + n^2)$;
extract $\sqrt{\ }$, $\quad x - y = \pm 2\sqrt{m^2 + n^2} \quad (3)$;
$\frac{1}{2}[(2) + (3)]$, $\quad x = m + n \pm \sqrt{m^2 + n^2}$;
$\frac{1}{2}[(2) - (3)]$, $\quad y = m + n \mp \sqrt{m^2 + n^2}$.

24. Let $h =$ the number of hours per day;
$n =$ " " " workmen.
Work $= 3nh = 6(n-4)(h-3) = 4(n+6)(h-6)$
$= 6(nh - 3n - 4h + 12) = 4(nh - 6n + 6h - 36)$;
(1), $\quad 0 = 3nh - 18n - 24h + 72 \quad (1)$;
(2), $\quad 0 = nh - 24n + 24h - 144 \quad (2)$;
(1) $\div 3$, $\quad nh - 6n - 8h + 24 \quad (3)$;
(3) $-$ (2), $\quad 0 = 18n - 32h + 168$;
$\div 2$, $\quad 0 = 9n - 16h + 84$;
$$n = \frac{16h - 84}{9} \quad (4);$$
$$nh = \frac{16h^2 - 84h}{9};$$
$$-24n = \frac{-24.16h + 24.84}{9};$$
$$0 = \frac{16h^2 - 84h}{9} + \frac{-24.16h + 24.84}{9} + 24h - 144;$$
$0 = 16h^2 - (84 + 24.16 - 9.24)h - 9.144 + 24.84$;
$0 = 16h^2 - 252h - 9.144 + 24.84$;
$0 = 4h^2 - 63h - 9.36 + 6.84$;

$$0 = 4h^2 - 63h + \frac{63^2}{16} = \frac{63^2}{16} + 9.36 - 6.84 = \frac{33^2}{16};$$

$$2h - \tfrac{63}{4} = \pm \tfrac{33}{4},$$

$$h = \frac{63 \pm 33}{8} = \frac{96 \text{ or } 30}{8} = 12 \text{ or } 3\tfrac{3}{4}.$$

Substituting the first value of h in (4), we have

$$n = \frac{192 - 84}{9} = 12.$$

Substituting the second value,

$$n = \frac{60 - 84}{9} = -\tfrac{8}{3},$$

a result algebraically correct, but practically inadmissible.

25. Let x be the amount by which the numbers differ from 9;

then $9 + x$ will be one,

and $9 - x$ the other.

$$(9 + x)^4 + (9 - x)^4 = 14\,096,$$

or $6561 + 2916x + 486x^2 + 36x^3 + x^4 + 6561 - 2916x + 486x^2 - 36x^3 + x^4 = 14\,096,$

or $13\,122 + 972x^2 + 2x^4 = 14\,096;$

transpose, $2x^4 + 972x^2 = 974;$

$\div 2,$ $x^4 + 486x^2 = 487;$

complete square,

$$x^4 + 486x^2 + (243)^2 = 59\,049 + 487 = 59\,536;$$

extract $\sqrt{\ }$, $x^2 + 243 = \pm 244;$

transpose, $x^2 = 1 \text{ or } -487,$

$$x = 1 \text{ or } \sqrt{-487};$$

then $9 + x = 10 \text{ or } 9 + \sqrt{-487},$

and $9 - x = 8 \text{ or } 9 - \sqrt{-487}.$

26. From the proportion,

$$19(x^3 + y^3) = 35(x^3 - y^3),$$

or $19x^3 + 19y^3 = 35x^3 - 35y^3;$

transpose, $54y^3 = 16x^3,$

$$27y^3 = 8x^3;$$

extract cube root, $3y = 2x$ (1).

From the equation, $x = \dfrac{24}{y}$ (2);

sub. in (1), $3y = \dfrac{48}{y};$

clear of fractions, $3y^2 = 48,$

$y^2 = 16, \ y = \pm 4;$

sub. in (2), $x = \dfrac{24}{\pm 4} = \pm 6.$

27. Let x be the amount by which the numbers differ from 7;

then $7 + x$ will be one,

and $7 - x$ the other.

$$(7+x)^5 + (7-x)^5 = 161\,294,$$

or $16\,807 + 12\,005\,x + 3430\,x^2 + 490\,x^3 + 35\,x^4 + x^5 + 16\,807 - 12\,005\,x + 3430\,x^2 - 490\,x^3 + 35\,x^4 - x^5 = 161\,294;$

transpose, $6860\,x^2 + 70\,x^4 = 127\,680;$

$\div$ 70, $x^4 + 98\,x^2 = 1824;$

complete square,

$$x^4 + 98\,x^2 + (49)^2 = 2401 + 1824 = 4225;$$

extract $\sqrt{\ }$, $x^2 + 49 = \pm 65,$

$x^2 = 16 \text{ or } -114,$

$x = 4 \text{ or } \sqrt{-114};$

then $7 - x = 3 \text{ or } 7 - \sqrt{-114},$

and $7 + x = 11 \text{ or } 7 + \sqrt{-114}.$

§ 208.

1. 7, 10, 13, 16. **2.** 11, 8, 5, 2, -1, -4, -7.

3. $a - 4n,\ a - 2n,\ a,\ a + 2n,\ a + 4n.$

§ 210.

2. First find first term,

$$a = b - (4-1)(-c) = b + 3c;$$

the series will be

$b + 3c,\ b + 2c,\ b + c,\ b,\ b - c,\ b - 2c,\ b - 3c.$

$\Sigma = 7b.$ Product, $b(b^2 - 9c^2)(b^2 - 4c^2)(b^2 - c^2).$

3. First find common difference,

$$d = \frac{a + 2b - (a + b)}{1} = b;$$

the first term will be $a + 2b - (4-1)b = a - b;$

the series will be $a - b,\ a,\ a + b,\ a + 2b,\ a + 3b.$

4. $d = \dfrac{9a + 7b - (a - b)}{8} = \dfrac{8a + 8b}{8} = a + b;$

second term is $a - b + a + b = 2a.$

5. $$\Sigma = n\frac{(a+l)}{2} \text{ or } \frac{a+l}{2} = \frac{\Sigma}{n};$$
$$\frac{a+l}{2} = \frac{108}{9} = 12, \quad a+l = 24.$$
One half of $a + l$ is the mean value, or 12.

6. $$d = \frac{9x - 9y - (7x - 5y)}{2} = \frac{2x - 4y}{2} = x - 2y;$$
$$a = 7x - 5y - (5-1)(x - 2y),$$
or $$7x - 5y - 4x + 8y = 3x + 3y;$$
the series will be
$3x + 3y$, $4x + y$, $5x - y$, $6x - 3y$, $7x - 5y$, $8x - 7y$, $9x - 9y$.

7. $$\Sigma = \frac{50\,(12 + 551)}{2} = 14\,075. \quad \text{Ans.}$$

8. $$\Sigma = 100 \cdot \frac{1 + 100}{2} = 50.101 = 5050. \quad \text{Ans.}$$

9. $$\Sigma = \frac{n\,(a+l)}{2} = \frac{n\,(1+n)}{2} = \frac{n+n^2}{2}.$$

10. $$\Sigma = \frac{n\,(1 + 2n - 1)}{2} = \frac{2n^2}{2} = n^2.$$

11. $$\Sigma = \frac{n\,(2 + 2n)}{2} = \frac{2n + 2n^2}{2} = n + n^2.$$

12. $$a = 134 - (m-1)\,6 = 140 - 6m;$$
$$\Sigma = \frac{m\,(134 + 140 - 6m)}{2} = \frac{274m - 6m^2}{2}, \text{ or}$$
$$137m - 3m^2.$$

13. $$n = \frac{2m - m + d}{d} = \frac{m+d}{d}.$$

14. $$d = \frac{10k - 1 - k}{8} = \frac{9k - 1}{8}.$$

15. 2 times the middle term is equal to the first and last $= 2s$;
$$\Sigma = \frac{n}{2} \times 2s = ns;$$

if the third term is s,

$$a = s - (3-1) - h = s + 2h,$$
$$l = 2s - (s + 2h) = s - 2h.$$

16. Let x be the middle term, and d the difference; then the series will be

$$x - 2d,\ x - d,\ x,\ x + d,\ x + 2d;$$

this sum or $\qquad 5x = 20,$

$$x = 4.$$

$$(x - 2d)^2 + (x - d)^2 + x^2 + (x + d)^2 + (x + 2d)^2 = 120;$$
$$x^2 - 4dx + 4d^2 + x^2 - 2dx + d^2 + x^2 + x^2 + 2dx + d^2$$
$$+ x^2 + 4dx + 4d^2 = 120;$$

collecting, $\qquad 5x^2 + 10d^2 = 120;$

$\div 5,$ $\qquad x^2 + 2d^2 = 24;$

substitute, $\qquad 16 + 2d^2 = 24;$

transpose, $\qquad 2d^2 = 8,$

$$d^2 = 4,$$
$$d = 2.$$

Terms, 0, 2, 4, 6, 8.

17. Let x be the second digit, and d the difference; then $\qquad x - d$ will be the first,

x " " second,

$x + d$ " " third;

the number will be

$$100(x - d) + 10x + x + d;$$

from the first condition,

$$x - d + x + x + d = 15, \text{ or } 3x = 15,\ x = 5;$$
$$100(x - d) + 10x + x + d - 792 = 100(x + d) + 10x$$
$$+ x - d,$$
$$\text{or } 100x - 100d + 10x + x + d - 792 = 100x + 100d$$
$$+ 10x + x - d;$$

transpose, $\qquad -198d = 792,$

$$d = -4;$$
$$x - d = 9,\ x + d = 1.$$

The number is 951.

18. Let x be the second term,

$$x - d = \text{the first},$$
$$x + d = \text{the third}.$$
$$x(x - d)(x + d) = 640,$$
$$x^3 - d^2x = 640 \ (1),$$
$$x + d = 4(x - d) = 4x - 4d,$$
$$5d = 3x,\ x = \tfrac{5}{3}d;$$

sub. in (1), $\dfrac{125\,d^3}{27} - \dfrac{5\,d^3}{3} = 640;$

clear of fractions, $125\,d^3 - 45\,d^3 = 17\,280,$

$$80\,d^3 = 17\,280,$$
$$d^3 = 216,$$
$$d = 6;$$
$$x = \frac{5\,d}{3} = 10,\; x - d = 4,$$
$$x + d = 16.$$

19. Clear of fractions the given equation,

$$264 = 54\,n - 3\,n^2 + 3\,n;$$

transpose, $3\,n^2 - 57\,n = -264;$

$\div 3,$ $n^2 - 19\,n = -88;$

complete sq., $n^2 - 19\,n + (\frac{19}{2})^2 = \frac{361}{4} - \frac{352}{4} = \frac{9}{4};$

extract $\sqrt{\ }$, $n - \frac{19}{2} = \pm\,\frac{3}{2},$

$$n = 11 \text{ or } 8.$$

Continuing the progression after the 9th term the terms will become negative, indicating a return of the traveller in the opposite direction, which will bring him back to his destination on the 11th day.

20
$$140 = 27\,n - \frac{3\,n\,(n-1)}{2};$$

clear of fractions, $280 = 54\,n - 3\,n^2 + 3\,n;$

transpose, $3\,n^2 - 57\,n = -280,$

$$n^2 - \tfrac{57}{3}\,n = -\tfrac{280}{3};$$

complete sq., $n^2 - \frac{57}{3}n + (\frac{57}{6})^2 = \frac{3249}{36} - \frac{3360}{36} = -\frac{111}{36};$

extract $\sqrt{\ }$, $n - \frac{57}{6} = \pm\sqrt{-\frac{111}{36}},$

which is imaginary. As the traveller, following the law of progression, would commence his return journey before reaching his destination, no answer is possible—a fact indicated by the imaginary result.

21. As in 19th, $160 = 25\,n - \dfrac{2\,n\,(n-1)}{2};$

clear of fractions, $320 = 50\,n - 2\,n^2 + 2\,n;$

transpose, $2\,n^2 - 52\,n = -320,$

$$n^2 - 26\,n = -160;$$

complete, $n^2 - 26\,n + 169 = 169 - 160 = 9;$

extract $\sqrt{\ }$, $n - 13 = \pm\,3,$

$$n = 13 \pm 3 = 16 \text{ or } 10.$$

22.
$$135 = 3\,n + \frac{3\,n\,(n-1)}{2};$$

clear of fractions, $270 = 6\,n + 3\,n^2 - 3\,n;$

transpose, $-3n^2 - 3n = -270,$
$$3n^2 + 3n = 270,$$
$$n^2 + n = 90;$$
complete, $n^2 + n + \frac{1}{4} = 90 + \frac{1}{4} = \frac{361}{4};$
extract $\sqrt{\ }$, $n + \frac{1}{2} = \pm \frac{19}{2},$
$$n = \pm \tfrac{19}{2} - \tfrac{1}{2} = 9 \text{ or } -10.$$

23. Let $x =$ 3d term;
then $x - d =$ 2d term;
$x - 2d =$ 1st "
$x + d =$ 4th "
$x + 2d =$ 5th "
$$(x-2d)(x-d) \times (x+d)(x+2d) = 12\,320,$$
or $x^5 - 5d^2x^3 + 4d^4x = 12\,320$ (1);
also, $x - 2d + x - d + x + x + d + x + 2d = 40;$
$$5x = 40,$$
$$x = 8;$$
sub. in (1), $32\,768 - 2560\,d^2 + 32\,d^4 = 12\,320;$
transpose, $32\,d^4 - 2560\,d^2 = -20\,448;$
$\div$ 32, $d^4 - 80\,d^2 = -639;$
complete, $d^4 - 80\,d^2 + 1600 = 1600 - 639 = 961;$
extract $\sqrt{\ }$, $d^2 - 40 = \pm 31,$
$$d^2 = 40 \pm 31 = 71 \text{ or } 9,$$
$$d = \sqrt{71} \text{ or } \pm 3.$$
The numbers are 2, 5, 8, 11, 14, or 14, 11, 8, 5, 2.

24. Clear of fractions, $q - p = -q + r;$
transpose, $2q = r + p,$
$$q = \frac{r+p}{2},$$
which shows that q is the mean.

25. Let x be the first term,
y the difference;
then $x + x + 2y + x + 4y + x + 6y + x + 8y = 90,$
or $5x + 20y = 90$, $x + 4y = 18$ (1),
and $x + y + x + 3y + x + 5y + 8 + 7y + x + 9y = 110,$
or $5x + 25y = 110$, $x + 5y = 22$ (2);
(2) $-$ (1), $y = 4$, $x = 18 - 4y = 2.$
The progression is 2, 6, 10, 14, 18, 22, 26, 30, 34, 38.

26. Let $2n + 1$ represent the number of terms;
then there will be $n + 1$ odd and n even terms.
Put S_1, the sum of the odd terms;
S_2, the sum of the even terms.

If x be the middle term,

$$x(n+1) = S_1, \quad \text{or} \quad nx + x = S_1 \ (1),$$

and $$nx = S_2 \ (2);$$

(1) − (2), $$x = S_1 - S_2,$$

$$n = \frac{S_2}{x} = \frac{S_2}{S_1 - S_2}.$$

As this is true for all values of d, d is indeterminate.

27. Let $2i$ = number of terms;
d = the common difference;
a = the first term of the odd series;
$a + d$ = the first term of the even series.

Because, in the series of odd terms, as well as in the series of even terms, each considered by itself, the number of terms is i and the common difference $2d$,

$$S_1 = \frac{2a + (i-1)2d}{2} i = 119 \ (1),$$

$$S_2 = \frac{2(a+d) + (i-1)2d}{2} i = 105 \ (2);$$

also, $l = a + (2i-1)d$, $(2i-1)d = l - a = 26$ (3);
eliminate a from (1) and (2);

(1) − (2), $$id = 14,$$

$$d = \frac{14}{i};$$

from (3), $$d = \frac{26}{2i-1} = \frac{14}{i};$$

clear of fractions, $$26i = 28i - 14,$$

$$2i = 14, \ i = 7, \ d = \frac{14}{i} = 2;$$

sub. in (1), $$a = 3.$$

∴ the series is 3, 5, 7, 9, 11, etc. etc. . . . 29.

28. If we wish i means there will be $i + 2$ terms in all; so that

$$d = \frac{l-a}{i+1}.$$

∴ the progression is a, $a + \dfrac{l-a}{i+1}$, $a + \dfrac{2(l-a)}{i+1}$, etc.

§ 211.

1. 1, 5, 25, 625, 3125.
2. 7, − 21, 63, − 189, 567.
3. 1, − 1, 1, − 1, 1.
4. $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{9}{32}$, $\frac{27}{128}$.
5. 4, $\frac{2}{5}$, $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{20}$.

§ 212.

1. From 3d term to 5th term inclusive are 3 terms:

$$a = 9,\ r = \tfrac{3}{2},\ n = 3.$$
$$l = 9 \times (\tfrac{3}{2})^2 = 9 \times \tfrac{9}{4} = \tfrac{81}{4}.$$

Now find 1st term, $l = \frac{81}{4}$, $n = 5$, $r = \frac{3}{2}$;

$$a = \frac{\frac{81}{4}}{(\frac{3}{2})^4} = \tfrac{81}{4} \times \tfrac{16}{81} = 4.$$

The series will be 4, 6, 9, $\frac{27}{2}$, $\frac{81}{4}$. Ans.

2. $a = \dfrac{\frac{32}{27}}{(-\frac{2}{3})^4} = \frac{32}{27} \times \frac{81}{16} = 6.$

The series will be 6, -4, $\frac{8}{3}$, $-\frac{16}{9}$, $\frac{32}{27}$. Ans.

3. $r = \left(\dfrac{l}{a}\right)^{\frac{1}{n-1}} = \left(\dfrac{x^4y^7}{y^4}\right)^{\frac{1}{4}} = x\sqrt[4]{y^3}.$

4. $r = \left(\dfrac{a^2}{1}\right)^{\frac{1}{3}} = a^{\frac{2}{3}};$

1, $a^{\frac{2}{3}}$, $a^{\frac{4}{3}}$, $a^{\frac{6}{3}}$.

5. $a = \dfrac{l}{r^{n-1}} = \dfrac{m}{-m} = -1;$

-1, m, $-m^2$, m^3.

6. $\Sigma = a\dfrac{r^n - 1}{r - 1} = 1 . \dfrac{2^{32} - 1}{1} = \$42,949,672.95.$

Last nail, 2^{31} cents $= \$21,474,836.48$.

7. $\Sigma = 2(2^{10} - 1) = 2(1024 - 1) = 2046.$

8. I. r^3. II. r^n.

9. If x is the required ratio,

$$x^2 = r^3,\ x = r^{\frac{3}{2}}.$$

10. Let the progression be

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6, \text{ etc.}$$

Then, by addition, we have

$$a + ar,\ ar + ar^2,\ ar^2 + ar^3,\ ar^3 + ar^4,\ ar^4 + ar^5, \text{ etc.};$$

factor, $a(1 + r)$, $ar(1 + r)$, $ar^2(1 + r)$, $ar^3(1 + r)$, etc.

This is a progression, since each term is r times the preceding term; likewise by subtracting we have

$$a(1 - r),\ ar(1 - r),\ ar^2(1 - r), \text{ etc.}$$

11. Let x be the given arithmetical mean of a and l, and y the given geometrical mean;

then
$$x = \frac{a + l}{2},$$
$$2x = a + l \quad (1);$$
$$y^2 = a \,.\, l,$$
$$a = \frac{y^2}{l} \quad (2);$$

sub. in (1), $\quad 2x = \frac{y^2}{l} + l;$

clear of fractions, $l^2 + y^2 = 2xl;$
transpose, $\quad l^2 - 2xl = -y^2;$
complete, $\quad l^2 - 2xl + x^2 = x^2 - y^2;$
extract $\sqrt{\ }$, $\quad l - x = \sqrt{x^2 - y^2},$

$$l = x + \sqrt{x^2 - y^2},$$

$$a = \frac{y^2}{l} = \frac{y^2}{x + \sqrt{x^2 - y^2}}.$$

12. $a + ar^3 : ar + ar^2 :: 21 : 5,$
$a(1 + r^3) : a(r + r^2) :: 21 : 5;$
$\div$ first ratio by a, $1 + r^3 : r + r^2 :: 21 : 5;$
$\div$ first ratio by $1 + r$, $1 - r + r^2 : r :: 21 : 5,$

$$5(1 - r + r^2) = 21r,$$

$$5 - 5r + 5r^2 = 21r;$$

transpose, $\quad 5r^2 - 26r = -5,$

$$r^2 - \tfrac{26}{5}r = -1;$$

complete, $\quad r^2 - \tfrac{26}{5}r + (\tfrac{13}{5})^2 = \tfrac{169}{25} - 1 = \tfrac{144}{25};$
extract $\sqrt{\ }$, $\quad r - \tfrac{13}{5} = \pm \tfrac{12}{5},$

$$r = \tfrac{13}{5} \pm \tfrac{12}{5} = 5 \text{ or } \tfrac{1}{5}.$$

13. $a, ar, ar^2, ar^3, \ldots\ ar^{n-1}.$

Multiplying, the exponent of a will be n.

The exponent of r will be the sum of the exponents from 1 to $n - 1$; so the product will be

$$a^n r^{1+2+3, \ldots + (n-1)} = a^n r^{\frac{n(n-1)}{2}}.$$

§ 214.

1. Limit of $\Sigma = \dfrac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{3} \times \frac{3}{2} = \frac{3}{6} = \frac{1}{2}.$

2. Limit of $\Sigma = \dfrac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{2}{3} \times \frac{3}{1} = \frac{6}{3} = 2.$

3. Limit of $\Sigma = \dfrac{\frac{1}{9}}{1 - (-\frac{1}{9})} = \dfrac{\frac{1}{9}}{1 + \frac{1}{9}} = \frac{1}{9} \times \frac{9}{10} = \frac{1}{10}.$

4. Limit of $\Sigma = \dfrac{\frac{4}{9}}{1 - \frac{4}{9}} = \frac{4}{9} \times \frac{9}{5} = \frac{4}{5}.$

§ 216.

1. $80 is the principal on which the company's interest is computed.

$$\Sigma = 80\,\frac{(1+.04)^{33} - (1+.04)}{.04};$$

$$\Sigma = 80\,\frac{3.64\,837 - 1.04}{.04};$$

$$\Sigma = 80\,\frac{2.60\,837}{.04} = 2000 \times 2.60\,837 = \$5216.74.$$

This sum, less the $5000 insurance, is $216.74, the gain for the company.

2. From the given equation $p = \dfrac{a}{\left(1 + \dfrac{c}{100}\right)^n}$.

3. The present worth of the first payment, by Problem II., gives

$$p = \frac{a}{1 + \frac{5}{100}}.$$

$$2\text{d} = \frac{a}{(1 + \frac{5}{100})^2}.$$

$$3\text{d} = \frac{a}{(1 + \frac{5}{100})^3}.$$ The total present worth

$$\frac{a}{1.05}\left(1 + \frac{1}{1.05} + \frac{1}{1.05^2}\right).$$

In parentheses is a geometrical progression, in which

$n = 3$, $r = \dfrac{1}{1.05}$, $a = 1$. Hence

$$\text{Value} = \frac{a}{1.05}\cdot\frac{1 - \dfrac{1}{1.05^3}}{1 - \dfrac{1}{1.05}} = \frac{a}{1.05}\cdot\frac{1.05^3 - 1}{1.05^3 - 1.05^2} =$$

$$\frac{a}{1.05^3}\cdot\frac{1.05^3 - 1}{1.05 - 1} = \frac{20\,a}{1.05^3}\,(1.05^3 - 1).$$

Computing by logarithms we find

$$2.72\,325\,a.$$

4. $p = \dfrac{a}{\left(1 + \dfrac{c}{100}\right)} =$ present value of 1st payment;

$$p = \frac{a}{\left(1+\frac{c}{100}\right)^2} = \text{present value of 2d payment;}$$

$$p = \frac{a}{\left(1+\frac{c}{100}\right)^3} = \text{" " " 3d "}$$

$$p = \frac{a}{\left(1+\frac{c}{100}\right)^n} = \text{" " " } n\text{th "}$$

The sum of present values will be

$$\frac{a}{\left(1+\frac{c}{100}\right)} + \frac{a}{\left(1+\frac{c}{100}\right)^2} + \frac{a}{\left(1+\frac{c}{100}\right)^3} \cdots\cdots + \frac{a}{\left(1+\frac{c}{100}\right)^n};$$

or, as this is a progression, it may be written

$$\frac{a\left[\left(1+\frac{c}{100}\right)^n - 1\right]}{\left(1+\frac{c}{100}\right)^{n+1} - \left(1+\frac{c}{100}\right)^n}.$$

If the first payment were due immediately,

$$p = \frac{a}{\left(1+\frac{c}{100}\right)^0} = a \text{ for 1st payment;}$$

$$p = \frac{a}{\left(1+\frac{c}{100}\right)} = a \text{ " 2d "}$$

$$p = \frac{a}{\left(1+\frac{c}{100}\right)^2} = a \text{ " 3d "}$$

$$p = \frac{a}{\left(1+\frac{b}{100}\right)^{n-1}} = a \text{ " } n\text{th "}$$

The sum would be

$$\frac{a}{\left(1+\frac{c}{100}\right)^0} + \frac{a}{\left(1+\frac{c}{100}\right)} + \frac{a}{\left(1+\frac{c}{100}\right)^2} \cdots\cdots + \frac{a}{\left(1+\frac{c}{100}\right)^{n-1}}.$$

As this is a progression, it may be written

$$\Sigma = \frac{a\left[\left(1+\frac{c}{100}\right)^n - 1\right]}{\left(1+\frac{c}{100}\right)^n - \left(1+\frac{c}{100}\right)^{n-1}}.$$

PART II.

ADVANCED COURSE.

§ 221.

1. Transpose, $u = 3 - \frac{3}{2}x$;
for $x = 1, 2, 3, 4, 5,$ etc.;
$u = 1\frac{1}{2},\ 0,\ -1\frac{1}{2},\ -3,\ -4\frac{1}{2},$ etc.

§ 222.

1. $$\begin{aligned} 2a + b &= c \ (1), \\ 5a - b &= c \ (2); \end{aligned}$$

$(1) + (2)$, $7a = 2c$,

$$a = \frac{2c}{7} \ (3);$$

$\times$ (1) by 5, $10a + 5b = 5c$ (4);
$\times$ (2) by 2, $10a - 2b = 2c$ (5);
$(4) - (5)$, $7b = 3c$,

$$b = \frac{3c}{7};$$

sub. in general equation, $ax + by = c$,
we get $\frac{2}{7}cx + \frac{3}{7}cy = c$;
cl. of fractions, $2cx + 3cy = 7c$;
$\div c$, $2x + 3y = 7$.

2. $$\begin{aligned} -2a - b &= c \ (1), \\ 2a + b &= c \ (2); \end{aligned}$$

$(2) - (1)$, $4a + 2b = 0$,

$$4a = -2b,$$

$$2a = -b,\ a = -\frac{b}{2};$$

$$ax - 2ay = 0;$$

$\div a$, $x - 2y = 0$.

3.

$$\begin{aligned}
& -5a + 2b = c \\
& \underline{5a - 2b = c} \\
(1) + (2), \quad & 0 = 2c \\
& \therefore c = 0; \\
& 5a - 2b = 0, \\
& b = \tfrac{5}{2}a; \\
& ax + \tfrac{5}{2}ay = 0; \\
\div \tfrac{1}{2}a, \quad & 2x + 5y = 0, \text{ the required equation.}
\end{aligned}$$

4.

$$\begin{aligned}
& 0a - 7b = c, \\
& 15a + 0b = c, \\
& -7b = c, \\
& b = -\frac{c}{7}; \\
& 15a = c, \\
& a = \frac{c}{15}; \\
\text{substitute,} \quad & \frac{c}{15}x - \frac{c}{7}y = c; \\
\text{clear of fractions,} \quad & 7cx - 15cy = 105c; \\
\div c, \quad & 7x - 15y = 105.
\end{aligned}$$

5.

$$\begin{aligned}
& 25a + 2b = c \ (1), \\
& 30a + 3b = c \ (2); \\
\times (1) \text{ by } 3, \quad & 75a + 6b = 3c \ (3); \\
\times (2) \text{ by } 2, \quad & 60a + 6b = 2c \ (4); \\
(3) - (4), \quad & 15a = c, \\
& a = \frac{c}{15}; \\
\times (1) \text{ by } 6, \quad & 150a + 12b = 6c \ (5); \\
\times (2) \text{ by } 5, \quad & 150a + 15b = 5c \ (6); \\
(6) - (5), \quad & 3b = -c, \\
& b = -\frac{c}{3}; \\
\text{substitute,} \quad & \frac{c}{15}x - \frac{c}{3}b = c; \\
\text{clear of fractions,} \quad & cx - 5cb = 15c; \\
\div c, \quad & x - 5b = 15.
\end{aligned}$$

§ 223.

1. If $u = 0$, $x = 3$ in (1);
$u = 1$, $x = 5$;
$u = 2$, $x = 7$;
etc. etc.

If $u = 0$, $x = 5$ in (2);
etc. etc.

§ 225.

1. $ay^2 - a^2y$.

2. $az^2 - a^2z$.

3. $ab^2y^2 - a^2by$.

4. $a(x+y)^2 - a^2(x+y)$.

5. $a(x+a)^2 - a^2(x+a) = ax^2 + a^2x$.

6. $a(x-a)^2 - a^2(x-a) = ax^2 - 3a^2x + 2a^3$.

7. $a(x+ay)^2 - a^2(x+ay)$.

8. $a(x-ay)^2 - a^2(x-ay)$.

9. $ax^4 - a^2x^2$.

10. ya^y.

11. $2ya^{2y}$.

12. $3ya^{3y}$.

13. $(x+y)a^{x+y}$.

14. $(x-y)a^{x-y}$.

15. a.

16. 1.

17. x^4.

18. x^6.

19. x^8.

20. x^{10}.

21. x^{2n}.

22. $\varphi(x+y) = a^{x+y} = a^x \times a^y$, and $\varphi(x) = a^x$, $\varphi(y) = a^y$; hence $\varphi(x) \times \varphi(y) = a^{x+y}$; $\therefore \varphi(x+y) = \varphi(x) \times \varphi(y)$; $\varphi(xy) = a^{xy} = (a^x)^y$; but $(a^x)^y = y$ power of $\varphi(x)$.

23. $6(6-1)(6-2)(6-3) = 360$.

24. $5(5-1)(5-2)(5-3) = 120$.

25. $4(4-1)(4-2)(4-3) = 24$.

26. $3(3-1)(3-2)(3-3) = 0$.

27. $2(2-1)(2-2)(2-3) = 0$.

28. $1(1-1)(1-2)(1-3) = 0$.

29. $0(0-1)(0-2)(0-3) = 0$.

30. $-1(-1-1)(-1-2)(-1-3) = 24$.

31. $-2(-2-1)(-2-2)(-2-3) = 120$.

§ 226.

1. $= 3y - 4x.$

2. $= 3a - 4b.$

3. $= 3.3 - 4.4 = -7.$

4. $= 3.4 - 4.3 = 0.$

5. $= 3.10 - 4.1 = 26.$

6. $= a^2 + b^2.$

7. $= ab + ab = 2ab.$

8. $= ay + bx.$

9. $= 7a - 3b.$

10. $= aq - pb.$

11. $= az + bx - aby.$

12. $= ab + ba - 2ab = 0.$

13. $= a^2 + b^2 - abc.$

14. $= a^3 + b^3 - abc^2.$

15. $= -a^2 - b^2 + a^2b^2.$

16. $= \dfrac{3(3-1)(3-2)}{3(3-1)(3-2)} = 1.$

17. $= \dfrac{4(4-1)(4-2)}{3(3-1)(3-2)} = 4.$

18. $= \dfrac{5(5-1)(5-2)}{3(3-1)(3-2)} = 10.$

19. $= \dfrac{6(6-1)(6-2)}{3(3-1)(3-2)} = 20.$

20. $= \dfrac{7(7-1)(7-2)}{3(3-1)(3-2)} = 35.$

21. $= \dfrac{8(8-1)(8-2)}{3(3-1)(3-2)} = 56.$

22. $= \dfrac{2(2-1)(2-2)}{-1(-1-1)(-1-2)} = 0.$

23. $= \dfrac{3(3-1)(3-2)}{-2(-2-1)(-2-2)} = -\frac{1}{4}.$

24. $= \dfrac{4(4-1)(4-2)}{-2(-2-1)(-2-2)} = -1.$

§ 226 (a).

1.

$$\begin{aligned}
a_0 &= 0\,(0+1) = 0 \\
a_1 &= 1\,(1+1) = 2 \\
a_2 &= 2\,(2+1) = 6 \\
a_3 &= 3\,(3+1) = 12 \\
a_4 &= 4\,(4+1) = 20 \\
a_5 &= 5\,(5+1) = 30 \\
a_6 &= 6\,(6+1) = 42 \\
a_7 &= 7\,(7+1) = 56 \\
a_8 &= 8\,(8+1) = 72 \\
a_9 &= 9\,(9+1) = 90 \\
a_{10} &= 10\,(10+1) = \underline{110} \\
\therefore a_0 + a_1 + \ldots + a_{10} &= 440
\end{aligned}$$

2.
$$a_1 = 1\,(1 + 1) = 2;$$
$$a_2 = 2\,(2 + 1) = 6;$$
$$a_1 + a_2 = 8;$$
$$a_2 = 2\,(2 + 1) = 6,\ \tfrac{4}{3}\,a_2 = 8;$$
$$\therefore a_1 + a_2 = \tfrac{4}{3}\,a_2.$$
$$a_1 = 2,\ a_2 = 6,\ a_3 = 12;$$
$$a_1 + a_2 + a_3 = 20,\ a_3 = 12,\ \tfrac{5}{3}\,a_3 = 20;$$
$\therefore$ the equation is true.
$$a_1 + a_2 + a_3 + a_4 = 40,\ a_4 = 20,\ \tfrac{6}{3}\,a_4 = 40;$$
$$\therefore \text{etc.}$$

3.
$$S_4 = 1 + 2 + 3 + 4 = 10;$$
$$S_5 = 1 + 2 + 3 + 4 + 5 = 15;$$
$$S_6 = 1 + 2 + 3 + 4 + 5 + 6 = 21;$$
$$S_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28;$$
$$\therefore S_4 + S_5 + S_6 + S_7 = 74.$$

4. In Example I. we found
$$a_4 = 20;$$
$$a_5 = 30;$$
$$a_6 = 42;$$
$$a_7 = 56;$$
$$\therefore a_4 + a_5 + a_6 + a_7 = 148.$$

5.
$$2\,S_5 = 30;$$
$$a_5 = 30;$$
$$2\,S_5 - a_5 = 0.$$

6.
$$2\,S_6 = 42;$$
$$a_6 = 42;$$
$$2\,S_6 - a_6 = 0.$$

§ 227.

1. If $A_{i+1} = A_i - A_{i-1}$.

Let $i = 1$;

then $A_{i+1} = A_2 = A_1 - A_0$.

When $i = 2,\ A_{i+1} = A_3 = A_2 - A_1 = -A_0$;

" $i = 3,\ A_{i+1} = A_4 = A_3 - A_2 = -A_1$;

" $i = 4,\ A_{i+1} = A_5 = A_4 - A_3 = -A_1 + A_0$;

" $i = 5,\ A_{i+1} = A_6 = A_5 - A_4 = A_0$;

" $i = 6,\ A_{i+1} = A_7 = A_6 - A_5 = A_1$;

" $i = 7,\ A_{i+1} = A_8 = A_7 - A_6 = A_1 - A_0$;

" $i = 8,\ A_{i+1} = A_9 = A_8 - A_7 = -A_0$;

" $i = 9,\ A_{i+1} = A_{10} = A_9 - A_8 = -A_1$.

Each subsequent term is equal to the third term preceding with its sign changed, or
$$A_n = -A_{n-3}.$$

2. If $A_{i+1} = 2A_i - A_0$:

When $i = 1$, $A_{i+1} = A_2 = 2A_1 - A_0$;

" $i = 2$, $A_{i+1} = A_3 = 2A_2 - A_0 = 4A_1 - 3A_0$;

" $i = 3$, $A_{i+1} = A_4 = 2A_3 - A_2 = 8A_1 - 7A_0$;

" $i = 4$, $A_{i+1} = A_5 = 2A_4 - A_3 = 16A_1 - 15A_0$.

3. If $A_{i+1} = iA_i + A_{i-1}$

When $i = 1$, $A_{i+1} = A_2 = A_1 + A_0$;

" $i = 2$, $A_{i+1} = A_3 = 2A_2 + A_1 = 3A_1 + 2A_0$;

" $i = 3$, $A_{i+1} = A_4 = 3A_3 + A_2 = 10A_1 + 7A_0$;

" $i = 4$, $A_{i+1} = A_5 = 4A_4 + A_3 = 43A_1 + 30A_0$.

4. If $A_i = A_{i-1} + h$,

$A_0 = A_0$.

When $i = 1$, $A_i = A_1 = A_0 + h$;

" $i = 2$, $A_i = A_2 = A_1 + h = A_0 + 2h$;

" $i = 3$, $A_i = A_3 = A_2 + h = A_0 + 3h$;

" $i = 4$, $A_i = A_4 = A_3 + h = A_0 + 4h$;

. . . .

" $i = n-1$, $A_i = A_{n-1} = A_{n-2} + h = A_0 + (n-1)h$;

" $i = n$, $A_i = A_n = A_{n-1} + h = A_0 + nh$;

$\therefore A_0 + A_1 + A_2 + A_3 \ldots A_n = A_0 + A_0 + h + A_0 + 2h \ldots A_0 + nh$.

Since the second member of this equation is an arithmetical progression of $n + 1$ terms, it may be written

$$(2A_0 + nh)\frac{n+1}{2}.$$

5. If $A_{i+1} = rA_i$:

When $i = 0$, $A_{i+1} = A_1 = rA_0 = rA_0$;

" $i = 1$, $A_{i+1} = A_2 = rA_1 = r^2A_0$;

" $i = 2$, $A_{i+1} = A_3 = rA_2 = r^3A_0$;

" $i = 3$, $A_{i+1} = A_4 = rA_3 = r^4A_0$;

. . . .

" $i = n - 1$, $A_{i+1} = A_n = rA_{n-1} = r^nA_0$;

$\therefore A_1 + A_2 + A_3 \ldots A_n = A_0(r + r^2 + r^3 \ldots r^n)$.

Since $r, r^2, r^3, \ldots$ etc., is a geometrical progression, the sum may be written $A_0\dfrac{r(r^n - 1)}{r - 1}$.

6. If $A_{i+1} = ikA_i + A_{i-1}$:

When $i = 1$, $A_{i+1} = A_2 = kA_1 + A_0 = kA_1 + A_0$;

" $i = 2$, $A_{i+1} = A_3 = 2kA_2 + A_1$

$= (2k^2 + 1)A_1 + 2kA_0$;

" $i = 3$, $A_{i+1} = A_4 = 3kA_3 + A_2$

$= (6k^3 + 4k)A_1 + (6k^2 + 1)A_0$;

When $i = 4$, $A_{i+1} = A_5 = 4kA_4 + A_3$
$= (24k^4 + 18k^2 + 1)A_1 + (24k^3 + 6k)A_0$;
" $i = 5$, $A_{i+1} = A_6 = 5kA_5 + A_4$
$= (120k^5 + 96k^3 + 9k)A_1 + (120k^4 + 36k^2 + 1)A_0$.

§ 228.

1. As this is the series of natural members, with n for the last term, there must be n terms, of which four are written; hence

$$n - 4 = \text{number omitted.}$$

2. Of $n - 2$ terms, 4 are written; hence

$$(n - 2) - 4 = n - 6 \text{ omitted.}$$

3. Of $n + 2$ terms, 4 are written; hence

$$(n + 2) - 4 = n - 2 \text{ terms omitted.}$$

4. In this series the quantity subtracted from n is less by 1 than the place it occupies; hence

$$n - s \text{ is the } (s + 1)\text{st term,}$$

or $s + 1$ represents the number of terms. Four terms being written, the number omitted is

$$s + 1 - 4 = s - 3.$$

5. We may reason in the following way: Since the terms form an arithmetical progression, the number omitted is one less than the difference between the two preceding and following the gap. This difference is

$$n - 2 - (n - s - 1) = s - 1.$$

Hence there are $s - 2$ omitted terms.

6. Reasoning in the same way we find s terms omitted.

7. $n + 1$. 8. $n - s + 1$. 9. $2k$.

NOTE. The above answers may also be obtained inductively by assigning special values, 2, 3, 4, or 5, to n or s, and comparing the results.

---

1. $= 1.2.3.4.5 = 120$. 2. $= 1.2.3.4.5.6 = 720$.

3. $= 1.2.3.4.5.6.7.8 = 40320$.

4. $= \dfrac{1.2.3.4.5.6.7}{1.2.3.1.2.3.4} = 35$.

5. $= \dfrac{1.2.3.4.5.6.7.8}{1.2.3.1.2.3.4.5} = 56$.

6. Factoring each term for 2, there being n terms, we may write the first member of the equation thus:

$$2^n(1.2.3.4.5....n) = 2^n.n!$$
$$\therefore 2.4.6.8....2n = 2^n.n!$$

7. $\frac{n}{2}! = 1.2.3.4....\frac{n}{2}$.

Multiply each of the $\frac{n}{2}$ factors in the second member by 2, and divide by the product of these factors, namely, $2^{\frac{n}{2}}$, and we have

$$\frac{n}{2}! = \frac{2.4.6.8.10....n}{2^{\frac{n}{2}}} = \frac{n(n-2)(n-4)....4.2}{2^{\frac{n}{2}}}.$$

§ 228, 2.

1. $\binom{8}{1} = 8$, $\binom{8}{2} = \frac{8.7}{1.2} = 28$;

$\binom{8}{3} = \frac{8.7.6}{2.3} = 56$, $\binom{8}{4} = \frac{8.7.6.5}{2.3.4} = 70$;

$\binom{8}{5} = \frac{8.7.6.5.4}{2.3.4.5} = 56$, $\binom{8}{6} = \frac{8.7.6.5.4.3}{2.3.4.5.6} = 28$;

$\binom{8}{7} = \frac{8.7.6.5.4.3.2}{2.3.4.5.6.7} = 8$, $\binom{8}{8} = \frac{8.7.6.5.4.3.2.1}{1.2.3.4.5.6.7.8} = 1$.

By adding $= 255$.

2. $\binom{3}{3} = \frac{3.2.1}{1.2.3} = 1$, $\binom{4}{3} = \frac{4.3.2}{1.2.3} = 4$;

$\binom{5}{3} = \frac{5.4.3}{1.2.3} = 10$, $\binom{6}{3} = \frac{6.5.4}{1.2.3} = 20$;

$\binom{7}{3} = \frac{7.6.5}{1.2.3} = 35$.

By adding $= 70$.

3. $\binom{5}{2} = \frac{5.4}{1.2}$. Multiplying both terms by $1.2.3$, the fraction becomes $\frac{5.4.3.2.1}{1.2.1.2.3} = \frac{5!}{2!\,3!}$

4. $\left(\frac{n}{s}\right) = \frac{n(n-1)(n-2)\ldots\ldots(n-s+1)}{1.2.3\ldots\ldots s}$.

Multiply both terms of the second member by $(n-s)! = 1.2.3\ldots\ldots n-s$. Because $n-s$ is the number next less than $n-s+1$, the numerator will become $1.2.3\ldots\ldots n$. Therefore

$$\left(\frac{n}{s}\right) = \frac{1.2.3\ldots\ldots n}{1.2.3\ldots\ldots s \times 1.2.3\ldots\ldots(n-s)}.$$

$$= \frac{n!}{s!\,(n-s)!}.$$

5. $\left(\frac{n+1}{s+1}\right) = \frac{(n+1)\{n(n-1)\ldots(n-s+1)\}}{(s+1)\{1.2.3\ldots\ldots s\}} = \frac{n+1}{s+1}\left(\frac{n}{s}\right)$.

6. $\left(\frac{n}{1}\right) = n,\ \left(\frac{n}{2}\right) = \frac{n(n-1)}{1.2};$

$$n + \frac{n(n-1)}{1.2} = \frac{2n+n(n-1)}{1.2}$$

$$= \frac{n[2+n-1]}{1.2} = \frac{n(n+1)}{1.2} = \left(\frac{n+1}{2}\right).$$

7. $\left(\frac{n}{2}\right) = \frac{n(n-1)}{1.2};$

$$\left(\frac{n}{3}\right) = \frac{n(n-1)(n-2)}{1.2.3}.$$

$$\frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} = \frac{3n(n-1)+n(n-1)(n-2)}{1.2.3}$$

$$= \frac{n(n-1)[3+n-2]}{1.2.3} = \frac{n(n-1)(n+1)}{1.2.3} = \left(\frac{n+1}{3}\right).$$

8. $\left(\frac{n}{3}\right) = \frac{n(n-1)(n-2)}{1.2.3};$

$$\left(\frac{n}{4}\right) = \frac{n(n-1)(n-2)(n-3)}{1.2.3.4};$$

$$\frac{n(n-1)(n-2)}{1.2.3} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} =$$

$$\frac{4n(n-1)(n-2)+n(n-1)(n-2)(n-3)}{1.2.3.4} =$$

$$\frac{n(n-1)(n-2)[4+n-3]}{1.2.3.4} = \frac{(n+1)n(n-1)(n-2)}{1.2.3.4}$$

$$= \left(\frac{n+1}{4}\right).$$

§ 230.

1. $= 2 \times 3 \times 2 \times 2 = 2^3 . 3.$

2. $= 3 \times 3 \times 2 \times 2 \times 2 = 3^2 . 2^3.$

3. $= 5 \times 2 \times 2 \times 13 = 5 . 2^2 . 13.$

4. $= 13 . 13 = 13^2.$ 5. $= 3 . 3 . 5 . 5 = 3^2 . 5^2.$

6. $= 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 = 2^8.$

7. $= 13 . 7.$ 8. $= 11 . 13.$

9. $= 3 . 3 . 2 . 2 . 2 . 5 = 3^2 . 2^3 . 5.$ 10. $= 7 . 31.$

11. $= 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 3 = 3 . 2^{10}.$

12. $= 2 . 3 . 2 . 2 . 5 . 2 . 3 . 7 . 2 . 2 . 2 . 3 . 3 = 2^7 . 3^4 . 5 . 7.$

§ 232.

2.

	Dividend.	Divisor.	Quo.	Rem.
	$427 =$	$399 \times$	$1 +$	28
	$399 =$	$28 \times$	$14 +$	7
	$28 =$	$7 \times$	4	

$\therefore 7 =$ G.C.D.

3.

$$\begin{aligned} 131 &= 91 \times 1 + 40 \\ 91 &= 40 \times 2 + 11 \\ 40 &= 11 \times 3 + 7 \\ 11 &= 7 \times 1 + 4 \\ 7 &= 4 \times 1 + 3 \\ 4 &= 3 \times 1 + 1 \\ 3 &= 1 \times 3 \end{aligned}$$

$\therefore 1 =$ G.C.D.

4.

$$\begin{aligned} 13 &= 8 \times 1 + 5 \\ 8 &= 5 \times 1 + 3 \\ 5 &= 3 \times 1 + 2 \\ 3 &= 2 \times 1 + 1 \\ 2 &= 1 \times 2 \end{aligned}$$

$\therefore 1 =$ G.C.D.

5.

$$\begin{aligned} 1000 &= 212 \times 4 + 152 \\ 212 &= 152 \times 1 + 60 \\ 152 &= 60 \times 2 + 32 \\ 60 &= 32 \times 1 + 28 \\ 32 &= 28 \times 1 + 4 \\ 28 &= 4 \times 7 \end{aligned}$$

$\therefore 4 =$ G.C.D.

	Dividend.		Divisor.		Quo.		Rem.
6.	1232	=	799	×	1	+	433
	799	=	433	×	1	+	366
	433	=	366	×	1	+	67
	366	=	67	×	5	+	31
	67	=	31	×	2	+	5
	31	=	5	×	6	+	1
	5	=	1	×	5		

$\therefore 1 = $ G.C.D.

$$\begin{aligned} \textbf{7.}\quad 1729 &= 800 \times 2 + 129 \\ 800 &= 129 \times 6 + 26 \\ 129 &= 26 \times 4 + 25 \\ 26 &= 25 \times 1 + 1 \\ 25 &= 1 \times 25 \end{aligned}$$

$\therefore 1 = $ G.C.D.

$$\begin{aligned} \textbf{8.}\quad 625 &= 250 \times 2 + 125 \\ 250 &= 125 \times 2 \end{aligned}$$

$\therefore 125 = $ G.C.D.

$$\begin{aligned} \textbf{9.}\quad 1000 &= 370 \times 2 + 260 \\ 370 &= 260 \times 1 + 110 \\ 260 &= 110 \times 2 + 40 \\ 110 &= 40 \times 2 + 30 \\ 40 &= 30 \times 1 + 10 \\ 30 &= 10 \times 3 \end{aligned}$$

$\therefore 10 = $ G.C.D.

10. If $n - p$ and n had any common factor, then, by § 231, Th. II., their difference, p, would have this same factor. But, by hypothesis, n and p have no common factor because they are prime to each other. Therefore $n - p$ and n can have no such factor, and are prime to each other, by definition.

11. By § 231, every divisor of both the numbers n and p is also a divisor of $n - p$. Also, every divisor of both n and $n - p$ is a divisor of their difference p. Therefore the common divisors of n and p are the same as the common divisors of n and $n - p$, wherefore the greatest of them must be the same.

We may also prove by applying Ex 10, because, if D is the G.C.D., $\frac{n}{D}$ and $\frac{p}{D}$ are prime to each other, whence $\frac{n}{D}$ and $\frac{n-p}{D}$ are also prime, by Ex. 10, showing D the G.C.D. of n and $n - p$.

12. Every divisor of $\frac{n+1}{2}$ must also be a divisor of its double, or of $n+1$. But no number can divide the two consecutive numbers n and $n+1$. Therefore no number which divides $\frac{n+1}{2}$ can divide n, and these numbers are prime, by definition.

§ 233.

1. Each tooth of the smaller will gear into four different teeth of the larger, and no more.

2. Dividing by G. C. D. $= 3$, 24 revolutions of small wheel $= 5$ of large wheel, when the same teeth will gear again. So each tooth of small wheel will gear with 24 of large, and each tooth of the large wheel with 5 of the small.

3. The ratio is $3:4$, so that each tooth of the small wheel will gear into 4 of the large one, and each tooth of the large one into 3 of the small one.

4 and 5. The numbers being prime to each other, every tooth of the one will gear into every tooth of the other.

§ 235.

1. Because the number leaves a remainder 1 when divided by 3, it is of the form $3n+1$; and because it is even, it is of the form $2m$. Therefore

$$3n + 1 = 2m.$$

The half of the number being m, let 3 go into m p times with the remainder r; then

$$m = 3p + r;$$
$$2m = 6p + 2r = 3n + 1;$$
$$2r - 1 = 3n - 6p.$$

Because the second member of the equation is a multiple of 3, so is the first member. Now r may have either of the three values 0, 1, or 2, and 2 is the only one of the three for which $2r - 1$ is a multiple of 3. Therefore it is the only remainder when m is divided by 3.

2. Let the number be

$$a10^3 + b10^2 + c10 + d.$$

Subtract, $ai^3 + bi^2 + ci + d$, and we have

$$a(10^3 - i^3) + b(10^2 - i^2) + c(10 - i).$$

This is divisible by $(10 - i)$ (§ 93).

§ 245.

1.
$$\begin{array}{llllll} a &= 3, & 7, & 16; & & \\ P &= 0 & 1 & 7 & 113 & \\ P' &= \overline{1}, & \overline{3}, & \overline{22}, & \overline{355}. & \text{Ans.} \end{array}$$

2.
$$\begin{array}{lllllll} a &= 3, & 2, & 2, & 3; & & \\ P &= 0 & 1 & 2 & 5 & 17 & \\ P' &= \overline{1}, & \overline{3}, & \overline{7}, & \overline{17}, & \overline{58}. & \text{Ans.} \end{array}$$

3.
$$\begin{array}{llllllll} a &= 3, & 1, & 3, & 1, & 3, & & \\ P &= 0 & 1 & 1 & 4 & 5 & 19 & \\ P' &= \overline{1}, & \overline{3}, & \overline{4}, & \overline{15}, & \overline{19}, & \overline{72}. & \text{Ans.} \end{array}$$

4.
$$\begin{array}{llllll} a &= 3, & 5, & x; & & \\ P &= 0 & 1 & 5 & 5x+1 & \\ P' &= \overline{1}, & \overline{3}, & \overline{16}, & \overline{16x+3}. & \text{Ans.} \end{array}$$

5.
$$\begin{array}{llllll} a &= a, & b, & c; & & \\ P &= 0 & 1 & b & cb+1 & \\ P' &= \overline{1}, & \overline{a}, & \overline{ba+1}, & \overline{c(ba+1)+a}. & \text{Ans.} \end{array}$$

§ 246.

1. $\frac{355}{113} = 3 + \frac{16}{113}$;
$\frac{113}{16} = 7 + \frac{1}{16}$; $\therefore$

$$\tfrac{113}{355} = \cfrac{1}{3 + \cfrac{1}{7 + \cfrac{1}{16}}}.$$

2. $1 \div \frac{1049}{3326} = \frac{3326}{1049} = 3 + \frac{179}{1049}$;
$\frac{1049}{179} = 5 + \frac{154}{179}$;
$\frac{179}{154} = 1 + \frac{25}{154}$;
$\frac{154}{25} = 6 + \frac{4}{25}$;
$\frac{25}{4} = 6 + \frac{1}{4}$; $\therefore$

$$\tfrac{1049}{3326} = \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{4}}}}}}$$

3. $1 \div \frac{628}{925} = \frac{925}{628} = 1 + \frac{297}{628};$

$$\frac{628}{297} = 2 + \frac{34}{297};$$
$$\frac{297}{34} = 8 + \frac{25}{34};$$
$$\frac{34}{25} = 1 + \frac{9}{25},$$
$$\frac{25}{9} = 2 + \frac{7}{9};$$
$$\frac{9}{7} = 1 + \frac{2}{7};$$
$$\frac{7}{2} = 3 + \frac{1}{2}. \quad \therefore$$

$$\frac{628}{925} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{8 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}}}}}}.$$

§ 248.

1. Because 1 is the greatest whole number in $\sqrt{3}$, we put

$$\sqrt{3} = 1 + \frac{1}{x}; \quad (1)$$

whence $$x = \frac{1}{\sqrt{3} - 1}.$$

Rationalizing the denominator, § 185,

$$2x = \sqrt{3} + 1.$$

Substituting for $\sqrt{3}$ its value in (1), we have

$$x = 1 + \frac{1}{2x}.$$

Putting this value of x in (1), and again in the denominator, and repeating the substitution indefinitely, we find

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1, \text{ etc.}}}}}}}}}}$$

Forming the convergents, we find them to be

$$\tfrac{1}{1},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{8}{11},\ \tfrac{11}{15},\ \tfrac{30}{41},\ \tfrac{41}{56}, \text{ etc.}$$

Adding unity to each of them, we find the approximate values of $\sqrt{3}$:

$$\tfrac{2}{1},\ \tfrac{5}{3},\ \tfrac{7}{4},\ \tfrac{19}{11},\ \tfrac{26}{15},\ \tfrac{71}{41},\ \tfrac{97}{56}, \text{ etc.}$$

2. $\sqrt{5} = 2 + \dfrac{1}{x}$; (1)

whence $x = \dfrac{1}{\sqrt{5} - 2}$.

Rationalizing the denominator, $x = \sqrt{5} + 2$.

Substituting for $\sqrt{5}$ from (1), we have $x = 4 + \dfrac{1}{x}$.

Putting this value of x in (1), and again in the denominator, and repeating the substitution indefinitely, we find

$$\sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4, \text{ etc.}}}}}}}}$$

Forming the convergents, we find them to be

$$\tfrac{1}{4},\ \tfrac{4}{17},\ \tfrac{17}{72},\ \tfrac{72}{305},\ \tfrac{305}{1292},\ \tfrac{1292}{5473}. \text{ etc.}$$

Adding 2 to each of them, we find the approximate values of $\sqrt{5}$:

$$\tfrac{9}{4},\ \tfrac{38}{17},\ \tfrac{161}{72},\ \tfrac{682}{305},\ \tfrac{2889}{1292},\ \tfrac{12238}{5473}, \text{ etc.}$$

3. $\sqrt{6} = 2 + \dfrac{1}{x}$; (1)

whence $x = \dfrac{1}{\sqrt{6} - 2}$.

Rationalizing denominator,

$$x = \frac{\sqrt{6} + 2}{2},\quad 2x = \sqrt{6} + 2.$$

From (1), $2x = 4 + \dfrac{1}{x}$, and $x = 2 + \dfrac{1}{2x}$.

Putting this value of x in (1), and repeating the substitution of x indefinitely, we find

$$\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2, \text{ etc.}}}}}}}}$$

Forming the convergents, we find them to be

$$\tfrac{1}{2},\ \tfrac{4}{9},\ \tfrac{9}{20},\ \tfrac{40}{89},\ \tfrac{89}{198}.\ \tfrac{396}{882}, \text{ etc.}$$

Adding 2 to each of them, we find the approximate values of $\sqrt{6}$:

$$\tfrac{5}{2},\ \tfrac{22}{9},\ \tfrac{49}{20},\ \tfrac{218}{89},\ \tfrac{485}{198},\ \tfrac{2160}{882}, \text{ etc.}$$

4. $\sqrt{10} = 3 + \dfrac{1}{x}$; (1)

whence $x = \dfrac{1}{\sqrt{10} - 3}$.

By rationalizing denominator,

$$x = \sqrt{10} + 3.$$

Substituting for $\sqrt{10}$ from (1), we have

$$x = 6 + \frac{1}{x}.$$

Putting this value of x in (1), as in previous examples, we find

$$\sqrt{10} = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6, \text{ etc.}}}}}}}$$

Forming the convergents, we find them to be

$$\tfrac{1}{6},\ \tfrac{6}{37},\ \tfrac{37}{228},\ \tfrac{228}{1405},\ \tfrac{1405}{8658},\ \tfrac{8658}{53\,353}, \text{ etc.}$$

Adding 3 to each, we find the approximate values of $\sqrt{10}$:

$$\tfrac{19}{6},\ \tfrac{117}{37},\ \tfrac{721}{228},\ \tfrac{4443}{1405},\ \tfrac{27\,379}{8658},\ \tfrac{168\,717}{53\,353}, \text{ etc.}$$

5. $\div\, x$ and we have $x - a = \dfrac{1}{x}$;

whence $x = a + \dfrac{1}{x}$.

Repeating this value of x indefinitely in the second member of the equation, and we have

$$x = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a, \text{ etc.}}}}}}}$$

Forming the convergents, we find them to be

$$\frac{1}{a}, \frac{a}{a^2+1}, \frac{a^2+1}{a^3+2a}, \frac{a^3+2a}{a^4+3a^2+1}, \frac{a^4+3a^2+1}{a^5+4a^3+3a},$$
$$\frac{a^5+4a^3+3a}{a^6+5a^4+6a^2+1}, \text{ etc.}$$

Adding a to each, we find the approximate values of x to be

$$\frac{a^2+1}{a}, \frac{a^3+2a}{a^2+1}, \frac{a^4+3a^2+1}{a^3+2a}, \frac{a^5+4a^3+3a}{a^4+3a^2+1},$$
$$\frac{a^6+5a^4+6a^2+1}{a^5+4a^3+3a}, \frac{a^7+6a^5+10a^3+4a}{a^6+5a^4+6a^2+1}, \text{ etc.}$$

§ 251.

1. (1) *bcd, bdc, cbd, cdb, dbc, dcb.*
(2) *acd, adc, cad, cda, dac, dca.*
(3) *abd, adb, bda, bad, dab, dba.*
(4) *abcd, abdc, acbd, acdb, adbc, adcb;*
bacd, badc, bcad, bcda, bdac, bdca;
cabd, cadb, cbad, cbda, cdab, cdba;
dabc, dacb, dbac, dbca, dcab, dcba.

2. The well-known English words are *mate, meat, tame,* and *team*—four in number. The total number of permutations being $P_4 = 4! = 24$, the proportion is $\frac{4}{24} = \frac{1}{6}$.

3. 1234, 1243, 1324, 1342, 1423, 1432;
2134, 2143, 2314, 2341, 2413, 2431;
3124, 3142, 3214, 3241, 3412, 3421;
4123, 4132, 4213, 4231, 4312, 4321.

4. Every different permutation of six different figures gives rise to a different number. Therefore the whole

number of different numbers which can be formed by permuting the six figures 1, 2, 3, 4, 5, 6 is

$$P_6 = 6\,! = 720.$$

5. Since the host and two of the guests are relatively fixed, there remain subject to permutation this group of three and the three remaining guests. Hence there are four things subject to permutation, and the total number of arrangements is

$$P_4 = 4\,! = 24.$$

6. The total number of different numbers that can be formed by permuting the seven first digits is

$$P_7 = 7\,! = 5040.$$

(*a*) Three of the digits are even and four are odd. Since any one digit will be at the end as often as another, $\frac{3}{7}$ of the numbers will terminate with an even digit and $\frac{4}{7}$ with an odd one. Therefore

total even numbers $= 5040 \times \frac{3}{7} = 2160;$
" odd " $= 5040 \times \frac{4}{7} = 2880.$

(*b*) Since there are four odd and only three even digits, a number in which the digits are alternately even and odd must begin and end with an odd digit.

A number may begin with any one of the four odd digits.

With whatever odd digit it begins, this digit may be followed by any of the three even ones.

This digit again may be followed by any one of the three remaining odd ones.

This again may be followed by any one of the two remaining odd ones. An even and an odd one will then be left for the last ones.

Therefore the total number of combinations subject to the condition of being alternately even and odd will be

$$4 \,.\, 3 \,.\, 3 \,.\, 2 \,.\, 2 = 144.$$

The result can also be reasoned out from the theory of permutations alone, as follows: Suppose any one of the required numbers to be written. The even numbers may stand in their places, and the four odd numbers be permuted among themselves in every way. We shall thus have $4\,! = 24$ different numbers.

Then in each of these numbers we may permute the three even digits in every possible way. We shall thus have six numbers from each of them, making the total number $6 \,.\, 24 = 144$.

(*c*) Consider the even numbers as a group or one thing; the four odd numbers as four additional things. Then for every possible permutation we can perform on these five things we shall have the group of three even numbers together.

The total number of these permutations is $P_5 = 120$.

Then in each of these arrangements we may permute the three even numbers in every way, and they will still be kept together.

Hence the total number of arrangements subject to the required condition is

$$P_3 \,.\, P_5 = 6 \,.\, 120 = 720.$$

(*d*) Reasoning in the same way, the four things composed of the three even digits and the odd digits as a separate group will admit of P_4 permutations.

In each of these permutations the odd numbers may be permuted in P_4 different ways without being separated.

Therefore the total number of the required permutations is $P_4^2 = 576$.

7. Since the letters *d*, *e*, *f* must stand in alphabetical order they admit of no permutations among themselves. Hence the total number of things which can be permuted are the five remaining letters and the group *def*, making six in all.

Hence the required number is

$$P_6 = 720.$$

8. In order that the word *deaf* may be found, the letters which compose it must stand together in alphabetical order. Hence the total number of things which can be permuted is made up of this word *deaf* and the remaining four letters, making five in all. Therefore the required number is $P_5 = 120$.

9. Taking out the words *age* and *bid* from the first nine letters, we have three letters left. Hence the total number of things to be permuted are these three letters, the word *age*, and the word *bid*, making five in all, and the answer is $P_5 = 120$.

10. Supposing the gentlemen to be arranged in numerical order, the first gentleman may choose any one of the five ladies, the second any one of the remaining four, etc.

Hence the total number of ways in which they can be divided into couples is

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

We may also reason by permutation, as follows:

If we suppose the ladies in any invariable order, we may permute the gentlemen in every possible way, and pair them with the ladies in the corresponding order. Therefore the total number of ways will be

$$P_5 = 5\,! = 120.$$

11. The condition requires that the group of letters *ade* shall be together. There being five letters in all, there will be three things to permute, *ade*, *b*, *c*. The number of these permutations is

$$P_3 = 3\,! = 6.$$

In each of these permutations we may change the group *ade* into *eda* and still fulfil the given condition.

Therefore the answer is

$$2 \cdot P_3 = 2 \cdot 3\,! = 12.$$

12. The two groups *abc* and *def* admit of two permutations as groups.

In each of these permutations the letters a, b, c may be permuted in every way without being separated, making six permutations. Similarly for the group *def*. Hence the required number is

$$2 \cdot P_3 \cdot P_3 = 72.$$

13. Since we are obliged to begin with the letters a, b, c, we have left four other letters to be permuted at pleasure.

We may permute these four letters in P_4 different ways.

For each of these permutations we may permute the letters a, b, c in every way. Hence the required number is $\quad P_3 \cdot P_4 = 144.$

14. Since the letters a, b, c are not required to come first, but may be permuted as a group with the four remaining letters, we shall have P_5 permutations in which these letters are together and in alphabetical order.

Then we may permute the three letters in every way in each of these permutations. Hence the required number is $\quad P_3 \cdot P_5 = 720.$

§ 252.

1. 35, 37, 39, 53, 57, 59, 73, 75, 79, 93, 95, 97.

2. 123, 124, 125, 132, 134, 135, 142, 143, 145, 152, 153, 154.

3. Total number of permutations is

$$n(n-1)(n-2)\ldots.(n-s+1)=$$
$$6(6-1)(6-2)\ldots.(6-4+1)=6.5.4.3=360.$$

4. The digit 1 may be followed by any combination of three out of the five remaining digits 2, 3, 4, 5, 6.

Hence the number that will begin with 1 is

$$5(5-1)(5-2)=60.$$

In the same way we may show that the digit 2 will lead in 60 permutations, and so for the six digits.

Therefore the total number of permutations is

$$6\times 60=360.$$

5. The gentlemen may select any permutation of three out of the five ladies.

Hence the number required is

$$5.4.3=60.$$

6. There are three even digits with which an even number formed in the required way may end, namely, 2, 4, and 6. Each of these even digits may be preceded by any permutations of two out of the remaining six digits. The number of these permutations is $6.5=30$. Hence the total number required is

$$3.30=90.$$

The question may also be reasoned out by showing that there are $7.6.5$ numbers of three digits, and that $\frac{3}{7}$ of these are even.

7. The required number may end with any one of the three even digits, each of these preceded by any one of the four odd digits. Each of these may be preceded by any one of the two remaining even ones. Hence the required number is

$$3.4.2=24.$$

§ 253.

1. $$n=7;$$
$$C_7=P_6=720.$$

2. Since the host is to have one particular guest on his right and another particular guest on his left, this group

of three cannot be permuted among themselves. The total number of objects to be permuted are therefore the five remaining guests and this group, making six. Hence the required number is

$$C_6 = P_5 = 120.$$

3. Considering the five works as five separate objects, they may be permuted on a circular shelf in

$$C_5 = P_4 = 24 \text{ different ways.}$$

Each work may then be arranged with the volumes in two different orders from right to left or from left to right. If it be required that this order shall be the same for all the volumes, the total number of arrangements will be $2 \,.\, 24 = 48.$

But if uniformity is not required in the arrangement, there will be twenty-five different orders for each permutation of the five works. The number required would then be $24 \,.\, 25 = 600.$

4. Considering b, a, d as one group, we shall have three things subject to the circular permutation. The number of permutations is $C_3 = 2$.

In each of these permutations the letters b and d may be interchanged and a still stand between them. Therefore the required number is

$$2 \,.\, 2 = 4.$$

5. The five even digits may be permuted around the circle in $C_5 = P_4 = 24$ different ways.

The five odd digits may then be permuted in every possible way among the five vacant places.

The number of these permutations is not C_5 but P_5, because if in any one arrangement the five odd numbers move to the right or left bodily the arrangement including all the numbers will be different. Hence for each circular permutation of these five odd digits there will be five different arrangements, making $5\,C_5 = P_5$. Hence the required number is

$$P_5 \,.\, C_5 = 5\,!\;\; 4\,! = 2880.$$

The same result may be reached by the following course of reasoning: Starting with any one of the ten digits, we may place next to it any one of the five of the opposite kind. Next to this we may place any one of the remaining four of the first kind. Next to this any one of the remaining four of the second kind, etc. Hence the total number of permutations will be

$$5 \,.\, 4 \,.\, 4 \,.\, 3 \,.\, 3 \,.\, 2 \,.\, 2 = 2880.$$

6. Considered as two groups of even and odd digits respectively, they admit of only one circular arrangement.

In this arrangement the even numbers may be permuted among themselves in P_5 different ways. In each of these arrangements the odd numbers may be permuted in P_5 different ways.

Hence the total number of arrangements subject to the given condition is

$$P_5^2 = 120^2 = 14\,400.$$

7. If the word *deaf* is to be spelled in but one direction, its letters will admit of no permutation among themselves. Hence the total number of things to be permuted will be three, and the number of circular permutations will be two.

If the word may be spelled in either direction, we may reverse the letters in each of these permutations, making four in all.

§ 254.

1. *aaab, aaba, abaa, baaa.*

2. *aabc, abac, abca, baca, bcaa, cbaa, caba, acba, acab, aacb, caab, baac.*

3. *aaabc, aaacb, aacab, acaab, caaab. caaba, cabaa, cbaaa, bcaaa, bacaa, baaca, baaac, abaac, aabac, aabca, abcaa, abaca, aacba, acaba, acbaa.*

4.
$$n = 7,\ r = 3,\ s = 3;$$
$$X_n = \frac{n\,!}{r\,!\,s\,!} = \frac{7\,!}{3\,!\,3\,!} = 140.$$

5.
$$n = 13,\ r = 4,\ s = 2,\ t = 2;$$
$$X_n = \frac{n\,!}{r\,!\,s\,!\,t\,!} = \frac{13\,!}{4\,!\,2\,!\,2\,!} = 64\,864\,800.$$

§ 255.

1. b is followed by 1 letter of lower order;

c	"	"	1	"	"
d	"	"	1	"	"
a	"	"	0	"	"
g	"	"	2	"	"
e	"	"	0	"	"

$\therefore$ number of inversions is 5.

2. Number of inversions is 5.

3. 3 is followed by 2 digits of lower order;
2 " " 1 " "
5 " " 2 " "
9 " " 2 " "
4 " " 1 " "
$\therefore$ number of inversions is 8.

4. Similar to Ex. 3. Number of inversions is 6.

5. Number of inversions is 16.

6. Number of inversions is 17.

§ 256.

1. By permuting the three symbols we have $a + b + c$, $a + c + b$, $b + a + c$, etc., which are all equal because the sum is the same in whatever order the numbers which form it are added.

2. Because the product is the same whatever the order of its factors, the product abc remains unchanged by a permutation of its factors.

3. By interchanging the factors a and b, the given expression becomes
$$b(a + c) + a(c + b) + c(b + a),$$
which we see to be identical with the given expression. In the same way making any other permutations we please among the letters, we shall find the expression to remain identically equal to that given.

4. The given expression may be put in the form
$$(1)\quad a^2b + b^2c + c^2a - (a^2c + b^2a + c^2b);$$
by interchanging a and b it will become
$$(2)\quad b^2a + a^2c + c^2b - (b^2c + a^2b + c^2a).$$
This is equal to the given expression with its sign changed.

REMARK. A function which changes its sign when two letters are interchanged is not a symmetric function, but is generally called an *alternating function*. By performing all possible permutations upon the letters which enter into it, we shall have two sets of values all equal in absolute value, but one half positive and the other half negative.

§ 257.

1. *ab, ac, ad, ae, bc, bd, be, cd, ce, de.*

2. *abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde.*

In writing the first set of combinations of two letters we leave behind a combination of three letters. Since these combinations of three letters are necessarily different, we must have at least as many combinations of three as of two.

We may show in the same way that there must be at least as many combinations of two as of three.

Hence the number of combinations of two and of three must be equal when the number of things is five.

This is expressed by the formula

$$C_s^n = C_{n-s}^n.$$

In this case $\qquad C_2^5 = C_{5-2}^5 = C_3^5.$

3. $C_s^n = \dfrac{n(n-1)(n-2)\ldots\ldots(n-s+1)}{s\,!} = \left(\dfrac{n}{s}\right).$

(1) $n = 3,\ s = 2;$

$$\therefore C_s^n = C_2^3 = \frac{3\,.\,2}{2\,!} = 3.$$

(2) $n = 7,\ s = 2;$

$$\therefore C_s^n = C_2^7 = \frac{7\,.\,6}{2\,!} = 21.$$

(3) $n = 9,\ s = 2;$

$$\therefore C_s^n = C_2^9 = \frac{9\,.\,8}{2\,!} = 36.$$

4. $C_s^n = \left(\dfrac{n}{s}\right).$

(1) $n = 4,\ s = 2;$

$$\therefore C_s^n = C_2^4 = \frac{4\,.\,3}{2\,!} = 6.$$

(2) $n = n,\ s = 2;$

$$\therefore C_s^n = C_2^n = \frac{n(n-1)}{2\,!} = \frac{n(n-1)}{2}.$$

If we draw a line from each point to every other point, we should draw every line twice, the two drawings being in opposite directions. Hence the actual number of different lines would be half of the $n(n-1)$ lines drawn.

5. $C_s^n = \left(\frac{n}{s}\right)$.

(1) The number of lines is five, and the intersection of any two of them gives a point.

Hence we have $n = 5,\ s = 2;$

$$\therefore C_s^n = C_2^5 = \frac{5 \cdot 4}{2\,!} = 10.$$

(2) $n = n,\ s = 2;$

$$\therefore C_s^n = C_2^n = \frac{n(n-1)}{2}.$$

6. Any combination of three out of the given n lines will form a triangle.

Since no two of the lines are coincident, these triangles will all be different.

A triangle can be formed only by some combination of three lines. Therefore the required number of triangles is equal to the number of combinations of three things in n.

Hence we have $n = n,\ s = 3;$

$$\therefore C_s^n = C_3^n = \frac{n(n-1)(n-2)}{3\,!}.$$

7. When two are paired, the other two may be paired to fulfil the conditions. Hence we have only half as many two pairs as combinations of two things in four. Therefore number of two pairs is

$$\frac{1}{2}C_2^4 = \frac{1}{2}\frac{4 \cdot 3}{1 \cdot 2} = 3.$$

8. Same exactly as Ex. 7.

9. (*a*) There are three things, namely, the three aces, and we have to take three at a time. Therefore number of ways $= C_3^3 = 1$.

(*b*) Since each die has the number 2, and they are to be taken in pairs, therefore we may have as many of the desired combinations as there are combinations of the three 2's taken two at a time, which is

$$C_2^3 = \frac{3 \cdot 2}{1 \cdot 2} = 3.$$

(*c*) The 1 of the first die may be combined with the 2 of the second die and 3 of the third, or with the 2 of the third die and the 3 of the second; thus making two

of the required combinations by beginning with the 1 of the first die. Similarly, by starting with the 1 of the second die we would have two more, and also two such combinations beginning with the 1 of the third die. Therefore in all we have six of the required combinations.

10. $n = 15,\ s = 5;$

$$\therefore C_s^n = C_5^{15} = \frac{15.14.13.12.11}{1.2.3.4.5} = 3003.$$

§ 258.

1. 111, 112, 113, 122, 123, 133, 222, 223, 233, 333.

Increasing second number by 1, and the third by 2, we have

123, 124, 125, 134, 135, 145, 234, 235, 245, 345,

which are the combinations of 1, 2, 3, 4, 5, taking 3 at a time.

2. $$R_s^n = C_s^{n+s-1} = \frac{n(n+1)(n+2)\dots(n+s-1)}{s\,!}.$$

(1) $n = 4,\ s = 4;$

$$\therefore R_s^n = R_4^4 = \frac{4.5.6.7}{1.2.3.4} = 35.$$

(2) $n = n,\ s = n;$

$$\therefore R_s^n = R_n^n = \frac{n(n+1)(n+2)\dots(n+n-1)}{1.2.3.4.5\dots n}$$

3. $n_1 = n - 1,\quad s = n + 1;$

$$\therefore R_s^{n_1} = R_{n+1}^{n-1} = C_{n+1}^{2n-1}$$

$$= \frac{(n-1)\,n\,(n+1)(n+2)\dots(2n-1)}{1.2.3.4\dots(n-1)\,n\,(n+1)}$$

$$= \frac{(n+2)(n+3)\dots(2n-1)}{1.2.3.4\dots(n-2)}.$$

§ 261.

1. First take a as the leading letter; then we may choose any two letters of the remaining four to complete the combination. Therefore

$$C_2^4 = \frac{4.3}{2} = 6 \text{ combinations beginning with } a.$$

Those beginning with b are followed by any two letters of the remaining three. The number of these pairs is

$$C_2^3 = \frac{3.2}{1.2} = 3.$$

To form those beginning with c and not containing a or b, c, can only be followed by the single combination of the two letters d and e. The number is

$$C_2^2 = \frac{2}{1.2} = 1.$$

2.
$$C_3^4 = \frac{4.3.2}{1.2.3};$$
$$C_2^4 = \frac{4.2}{1.2};$$
multiplying by $\frac{3}{3}$, we have
$$C_2^4 = \frac{4.3}{1.2} \times \frac{3}{3} = \frac{4.3.3}{1.2.3}.$$
$$\therefore C_3^4 + C_2^4 = \frac{4.3\,(2+3)}{1.2.3} = \frac{5.4.3}{1.2.3},$$
and
$$C_3^5 = \frac{5.4.3}{1.2.3}.$$
Hence
$$C_3^5 = C_3^4 + C_2^4.$$

(1) The proof as suggested in the text.

(2) $C_s^{n+1} = C_s^n + C_{s-1}^n$.

To prove:
$$\left(\frac{n+1}{s}\right) = \left(\frac{n}{s}\right) + \left(\frac{n}{s-1}\right);$$
$$\left(\frac{n}{s}\right) = \frac{n\,(n-1)\,(n-2)\,(n-3)\ldots(n-s+1)}{s\,!}; \quad (a)$$
$$\left(\frac{n}{s-1}\right) = \frac{n\,(n-1)\,(n-2)\ldots(n-s+2)}{(s-1)\,!}.$$
Multiplying this last expression by $\frac{s}{s}$, we have
$$\left(\frac{n}{s-1}\right) = \frac{n\,(n-1)\,(n-2)\ldots(n-s+2)\,s}{s\,!}. \quad (b)$$
Adding (a) and (b), we have
$$\frac{n\,(n-1)\,(n-2)\ldots(n-s+2)\,\{(n-s+1)+s\}}{s\,!}$$
$$= \frac{(n+1)\,n\,(n-1)\,(n-2)\ldots(n-s+2)}{s\,!};$$

$$\binom{n+1}{s} = \frac{(n+1)\,n\,(n-1)\cdot(n-2)\ldots(n-s+2)}{s\,!}$$
$$= (a) + (b);$$
$$\therefore \binom{n+1}{s} = \binom{n}{s} + \binom{n}{s-1}.$$

3. The number 7 may be combined with any two of the remaining six. Therefore the number which contain 7 will be $C_2^6 = \dfrac{6\,.\,5}{1\,.\,2} = 15.$

The number which will not contain 7 will be the combinations of 3 out of 1, 2, 3, 4, 5, 6; the number of which is $C_3^6 = \dfrac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3} = 20.$

4. The letter a may be combined with any combination of $s-1$ letters out of the $n-1$ letters $b, c, \ldots n$.

Therefore the number which contain a is

$$C_{s-1}^{n-1} = \binom{n-1}{s-1}.$$

The combinations which do not contain a will be the combinations of s letters out of the $n-1$ letters $b, c, \ldots n$.

Therefore the number which do not contain a is

$$C_s^{n-1} = \binom{n-1}{s}.$$

NOTE. By taking the sum of these quantities we have for the total number of combinations

$$C_s^n = C_{s-1}^{n-1} + C_s^{n-1}.$$

The accordance of this result with that of Ex. 2 should be understood by the student.

5. The combination abc may be combined with any $s-3$ letters out of the $n-3$ letters which remain after taking away abc.

Therefore the required number is

$$C_{s-3}^{n-3} = \binom{n-3}{s-3}.$$

§ 263.

1. Since any one of the ten horses forming the first stud may be combined with any one of the twelve forming the second, the total number of combinations is

$$10 \times 12 = 120.$$

2. Since s may be followed by either ch, c, or k, there is here a choice of three combinations.

Next we have a choice of four, making with the first choice $3 \cdot 4$ or 12 combinations.

Similarly for the others.

Therefore the total possible number of ways is found to be $3 \cdot 4 \cdot 2 \cdot 5 \cdot 2 = 240.$

3. Two ways may occur each time it is thrown. Therefore in two throws we would have

$$2 \cdot 2 = 2^2 \text{ ways,}$$

and in n throws we would have

$$2^n \text{ different ways.}$$

4. Similar to Ex. 3.

$$n = 4,\ a = b = c = d = 3;$$
$$\therefore \text{ number of routes} = 3^4 = 81.$$

§ 266.

1. Since one side is as likely to come up as another and the total number of sides is 6, this is the total number of cases.

Two of these cases leading to white and four to black, we have

$$\text{Probability of a white side } \tfrac{2}{6} = \tfrac{1}{3};$$
$$\text{" \quad " \quad black \quad " } \tfrac{4}{6} = \tfrac{2}{3}.$$

2. When n is odd, the number of black balls is

$$\frac{n+1}{2}.$$

Therefore probability of drawing a black ball is

$$\frac{\frac{n+1}{2}}{n} = \frac{n+1}{2n}.$$

When n is even, the number of black balls is

$$\frac{n}{2}.$$

Therefore probability of drawing a black one is

$$\frac{\frac{n}{2}}{n} = \frac{1}{2}.$$

3. Number of white balls $= n + 1$;

" red " $= n + 1$;

" black " $= n$.

$$\therefore \text{ probability of a white ball is } \frac{n+1}{3n+2};$$
$$\text{" " red " } \frac{n+1}{3n+2};$$
$$\text{" " black " } \frac{n}{3n+2}.$$

§ 267.

1. Since there are four keys and any three may be taken, the number of cases is

$$C_3^4 = \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} = 4.$$

The number of cases the two safe-keys may be taken is the number of combinations of the two safe-keys with one of the other two, which is

$$C_1^2 = 2;$$
$$\therefore \text{ probability is } \tfrac{2}{4} = \tfrac{1}{2}.$$

2. The number of cases is

$$P_3 = 1 \cdot 2 \cdot 3 = 6.$$

(*a*) The number of times the brothers may sit together is equal to the number of permutations of the brothers as a group with the third person, which is $P_2 = 1 \cdot 2 = 2$, multiplied by the number of permutations of the brothers among themselves, which is $P_2 = 1 \cdot 2 = 2$. Hence the total number of times the brothers may sit together is $2 \cdot 2 = 4$.

Therefore probability of the brothers sitting together is $\tfrac{4}{6} = \tfrac{2}{3}$.

(*b*) The third man may only be between the brothers twice, which is the number of times the brothers may be permuted, the third man remaining fixed;

$$\therefore \text{ probability is } \tfrac{2}{6} = \tfrac{1}{3}.$$

3. By Prob. 3 the total number of cases is $6^2 = 36$.

(*a*) There is but one case for both aces;

$$\therefore \text{ probability is } \tfrac{1}{36}.$$

(*b*) The ace side of one die may be combined with any one of the five sides of the other die; the same for the ace side of the other die with the five sides of the first die,—making ten cases;

$$\therefore \text{ probability is } \tfrac{10}{36} = \tfrac{5}{18}.$$

4. As before, the total number of cases is 36.

The maximum number of one face being 6, the number 8 may be formed by $2+6$, $3+5$, $4+4$, $5+3$, or $6+2$, giving five combinations.

Therefore the required probability is $\frac{5}{36}$.

5. A party of thirteen persons may seat themselves around a table in $C_{13} = P_{12}$ different possible ways.

This is the total number of cases.

The cases which bring Mr. Taylor and Mr. Williams together will be found by permuting the group composed of these two gentlemen in every possible way with the eleven remaining persons.

The number of these permutations is

$$C_{12} = P_{11}.$$

But Mr. Taylor and Mr. Williams may change places and be still together, so that there are $2\,P_{11}$ arrangements in which they are together.

Therefore the required probability is

$$\frac{2\,P_{11}}{P_{12}} = \frac{2\,.\,1\,.\,2\,.\,3\,.\,.\,.\,.\,11}{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,.\,12} = \frac{2}{12} = \frac{1}{6}$$

6. There being four volumes in all, the total number of ways in which the servant may arrange them is 24. The two groups of two volumes each may change places and in each position the two volumes may interchange and still be kept together.

Therefore the total number of permutations in which each set is together is 8.

The required probability is therefore

$$\tfrac{8}{24} = \tfrac{1}{3}.$$

7. There being six individual groups in the closet, the valet may choose any combination of two.

The total number of these combinations is

$$C_2^6 = 15.$$

To form a combination of right and left, any one of the three rights may be combined with any one of the three lefts, making three square combinations. Therefore the probability that it is right and the other left is

$$\tfrac{9}{15} = \tfrac{3}{5}.$$

There being three pairs of boots, there will be three combinations out of the fifteen belonging to the pair.

Therefore this probability is

$$\tfrac{3}{15} = \tfrac{1}{5}.$$

Note. This last result may also be right by the consideration that after picking up one boot he will have five boots left, one of which will be the mate to that picked up. The probability of his selecting this mate is $\frac{1}{5}$.

8. Total number of ways in which they may be placed is

$$P_3 = 1 . 2 . 3 = 6.$$

Only two would read in order in either direction;

$$\therefore \text{ probability is } \tfrac{2}{6} = \tfrac{1}{3}.$$

9. The total number of spans among the eight horses is

$$C_2^8 = \frac{8 . 7}{1 . 2} = 28.$$

From the five horses taken at random $C_2^5 = 10$ spans may be selected.

Therefore the required probability is

$$\tfrac{10}{28} = \tfrac{5}{14}.$$

10. The three digits drawn may be any one of the

$$C_3^5 = \frac{5 . 4 . 3}{1 . 2 . 3} = 10 \text{ combinations.}$$

A combination containing 2 and 5 can have but one of the three remaining numbers combined with it.

Therefore the total number of these combinations is

$$C_1^3 = 3.$$

The required probability is therefore $\frac{3}{10}$.

11. $1 + 5$ and $2 + 4$ are the only two whose sum is 6. There is only one chance for each, or two chances in all;

$$\therefore \text{ probability is } \tfrac{2}{10} = \tfrac{1}{5}.$$

12. Since each purse contains only one kind of coin, the probability of selecting an eagle is simply that of selecting the purse which contains eagles.

One purse is as likely to be selected as another; the probability of selecting each is $\frac{1}{3}$;

$\therefore$ probability of selecting an eagle is $\frac{1}{3}$.

13. There are seven balls in all, and two are to be drawn. Therefore total number of cases is

$$C_2^7 = \frac{7 . 6}{1 . 2} = 21.$$

Number of cases when both are white is

$$C_2^3 = \frac{3 . 2}{1 . 2} = 3.$$

Number of cases when both are black is

$$C_2^4 = \frac{4 \cdot 3}{1 \cdot 2} = 6.$$

$3 + 6 = 9$ cases when both are of same color;

$\therefore$ required probability is $\frac{9}{21} = \frac{3}{7}$.

14. Since there are two chances for the better player to one for the poorer one, we may suppose three cases to each game, two of which are in favor of the better player. Any case of the first game may be combined with one of the second, and this again with any case of the third, making the number of combinations

$$3 \cdot 3 \cdot 3 = 27.$$

The weaker opponent's chances are those of winning games $1 \cdot 2$, $1 \cdot 3$, $2 \cdot 3$, $1 \cdot 2 \cdot 3$. The number of this is 4.

Therefore his total probability is $\frac{4}{27}$.

15. The total number of possible combinations is

$$C_2^{m+n} = \frac{(m+n) \times (m+n-1)}{2}.$$

A combination of one white and one black ball may be formed of any one of the n white balls combined with any one of the m black ones, making mn in all.

The quotient of these numbers is

$\dfrac{2\,mn}{(m+n)\,(m+n-1)}$, the required probability.

16. There being six balls in all, of which three are drawn, the total number of cases is

$$C_3^6 = 20.$$

In order that the three may be of different colors, the one white ball may be combined with either of the two red ones, and each of these with either of the three black ones. The number of these combinations is 6;

$\therefore$ the probability is $\frac{6}{20} = \frac{3}{10}$.

17. By § 267, Prob. 2, the number of cases is 2^n, of which there are n cases of one bead and no more;

$$\therefore \text{probability is } \frac{n}{2^n}.$$

18. The total number of sub-committees is

$$C_3^{11} = \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} = 165.$$

Any two of the six Republicans may be combined with any one of the five Democrats.

The number of combinations of the required kind is

$$5 \,.\, C_2^6 = 5\,\frac{6 \,.\, 5}{1 \,.\, 2} = 75;$$

$$\therefore \text{ probability is } \tfrac{75}{165} = \tfrac{5}{11}.$$

§ 269.

1. The probability that the first will succeed is $\frac{3}{5}$, and that the second will succeed is $\frac{2}{3}$.

(1) Therefore the probability that both will succeed is

$$\tfrac{3}{5} \cdot \tfrac{2}{3} = \tfrac{6}{15} = \tfrac{2}{5}.$$

In the same way we find that the probability of the first one failing is $\frac{2}{5}$, and of the second failing $\frac{1}{3}$.

(2) Therefore the probability that both will fail is

$$\tfrac{1}{3} \times \tfrac{2}{5} = \tfrac{2}{15}.$$

2. Probability of man living is $\frac{7}{8}$.

" wife " $\frac{9}{10}$.

(α) Probability of both living is $\frac{7}{8} \cdot \frac{9}{10} = \frac{63}{80}$.

(β) Probability both are dead is $\frac{1}{8} \cdot \frac{1}{10} = \frac{1}{80}$.

(γ) Probability of husband living but wife dead is

$$\tfrac{7}{8} \cdot \tfrac{1}{10} = \tfrac{7}{80}.$$

(δ) Probability of husband dead but wife living is

$$\tfrac{1}{8} \cdot \tfrac{9}{10} = \tfrac{9}{80}.$$

3. In order that a man may be unable to go through a door it must be locked, and the two keys which he has must neither of them be the right one. The first probability is $\frac{2}{3}$ and the second is $\frac{1}{2}$.

Therefore the probability that he will be unable to go through is $\frac{1}{3}$.

Subtracting this from unity, we find $\frac{2}{3}$ to be the probability that he can go through.

4. By the theorem the probability that both balls will be white is equal to the product of the probability that the first one drawn will be white multiplied by the probability that in case this is white the second one will be white also. The probability that the first ball drawn is white is $\frac{3}{7}$. If he draws this ball and puts it into the other bag, which then contains an equal number of white and black balls, the probability that the second ball drawn is white is then $\frac{1}{2}$.

Hence the probability of two drawings of the white ball is $\frac{3}{7} \times \frac{1}{2} = \frac{3}{14}$.

5. The total number of possible committees will be C_3^{n+m}. A committee consisting of two Democrats and one Republican may be formed of any one of the C_2^m combinations of two Democrats with any one of the n Republicans, of which the number is $n \,.\, C_2^m$.

The quotient of these two quantities is

$$\frac{3n \,.\, m(m-1)}{(m+n)(m+n-1)(m+n-2)},$$

the required probability.

6. There being but one prize, no one can gain it unless all who precede it may fail.

There being seven balls, A's chance is $\frac{2}{7}$.

If A fails, he must have drawn one of the five black balls, leaving two white and four black ones. The probability of his doing this is $\frac{5}{7}$. Supposing this to happen, B, having to draw from six balls, will have a chance of $\frac{2}{6} = \frac{1}{3}$.

$\frac{1}{3} \,.\, \frac{5}{7} = \frac{5}{21}$, the probability of B winning.

In order that C may succeed, A and B must both fail. The probability of A's failing is $\frac{5}{7}$, and in case he fails the probability of B failing will be $\frac{2}{3}$, because he draws from two white and four black balls; and the probability of C succeeding will be $\frac{2}{5}$ if A and B fail. Therefore C's probability of success is

$$\frac{5}{7} \cdot \frac{2}{3} \cdot \frac{2}{5} = \frac{4}{21}.$$

In order that D may succeed, A, B, and C must all fail, and in this case he will draw from four balls, of which two are white, giving him a probability of $\frac{1}{2}$.

Reasoning as before, we find the product of the several numbers which lead to his success to be

$$\frac{5}{7} \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{2} = \frac{3}{21} = \frac{1}{7}.$$

In order that E may succeed, A, B, C, and D must all fail, and if this happens he will have two white balls and one black to draw from, giving him a probability of $\frac{2}{3}$.

Reasoning as before, we find the probability to be

$$\frac{5}{7} \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{21}.$$

In order that F may succeed, the five preceding ones must all fail, and then he will be certain of success because only the two white balls will be left.

The chances we find to be

$$\frac{5}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{21}.$$

7. A's probability of winning is evidently $\frac{1}{2}$. If A fails, then B has a probability of $\frac{1}{2}$.

Since B's success requires the concurrence of two events, each of probability $\frac{1}{2}$, his probability is

$$\tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}.$$

The remaining probability is $\frac{1}{4}$, which is X's chance for retaining the prize.

8. A has a probability of $\frac{1}{2}$ of winning on the first throw, and the probability of $\frac{1}{2}$ of losing.

If he fails, but not otherwise, B has a probability of $\frac{1}{2}$ of winning.

If B fails, but not otherwise, A has a single throw with the probability of $\frac{1}{2}$.

Multiplying the several probabilities we find:

$\frac{1}{2}$ that A wins on his first throw;

$\frac{1}{2^2}$ " B " " " "

$\frac{1}{2^3}$ " A " " second "

$\frac{1}{2^4}$ " B " " " "

Adding up the several probabilities, they are:

$$\text{A's} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \text{etc. ad. in.} = \frac{2}{3}.$$

$$\text{B's} = \frac{1}{2^2} + \frac{1}{2^4} + \text{etc. ad. in.} = \frac{1}{3}.$$

The solution may also be reached by the following more elegant process.

Let x be the unknown probability that A will win.

If A fails on the first throw, B will have exactly the same chance that A had before he began; that is, B's probability of winning will in this case be x.

But there is a probability of $\frac{1}{2}$ that A will win on the first throw, and that B will have a chance at the prize. Therefore before the throws begin B's chance is $\frac{1}{2}x$.

Therefore the sum of the two probabilities is

$$x + \tfrac{1}{2}x = \tfrac{3}{2}x.$$

Since one or the other must ultimately win, this sum is unity. Therefore

$$\tfrac{3}{2}x = 1;$$

$$x = \tfrac{2}{3}.$$

$$\text{B's probability} = \tfrac{1}{2}x = \tfrac{1}{3}.$$

9. Reasoning as in the last example, and remembering that if the first man fails on the first throw he will have no other chance unless the remaining $n - 1$ men fail, we find his total probability to be

$$\frac{1}{2} + \frac{1}{2^{n+1}} + \frac{1}{2^{2n+1}} + \text{etc.} = \frac{1}{2}\left(1 + \frac{1}{2^n} + \frac{1}{2^{2n}}, \text{ etc.}\right).$$

In the same way we find the chances of the second man to be

$$\frac{1}{2^2} + \frac{1}{2^{n+2}} + \frac{1}{2^{2n+2}} + \text{etc.} = \frac{1}{4}\left(1 + \frac{1}{2^n} + \frac{1}{2^{2n}}, \text{ etc.}\right).$$

The sum of the geometrical progression in the second member is

$$\frac{2^n}{2^n - 1}.$$

Therefore the several probabilities of winning are:

$$\text{A's} = \frac{1}{2}\frac{2^n}{2^n - 1};$$

$$\text{B's} = \frac{1}{4}\frac{2^n}{2^n - 1}.$$

10. The probability that the first man will win on the first throw is $\frac{1}{6}$. If he fails, he has no other chance unless the two following men fail. The probability that these men will both fail is

$$\frac{5^2}{6^2} = \frac{25}{36}.$$

Therefore his chance of winning on a second trial is

$$\frac{1}{6} \cdot \frac{5^3}{6^3}.$$

Summing up his chances of winning on the first, second, and third trial, etc., we find it to be

$$\frac{1}{6}\left(1 + \frac{5^3}{6^3} + \frac{5^6}{6^6} + \frac{5^9}{6^9} + \text{etc.}\right) = \frac{1}{6} \cdot \frac{216}{91} = \frac{36}{91}.$$

Summing the chances of the second man in the same way and remembering that he has no throw unless the first man fails, of which the probability is $\frac{5}{6}$, we find his total probability to be

$$\frac{5}{6} \cdot \frac{1}{6}\left(1 + \frac{5^3}{6^3} + \frac{5^6}{6^6} + \text{etc.}\right) = \frac{5}{36} \cdot \frac{216}{91} = \frac{30}{91}.$$

The third man has no chance unless the two preceding fail.

Summing up his several chances of winning on the first, second, third, etc., trials, we find them to be

$$\frac{5^2}{6^2} \cdot \frac{1}{6}\left(1 + \frac{5^3}{6^3} + \frac{5^6}{6^6} + \text{etc.}\right) = \frac{25}{216} \cdot \frac{216}{91} = \frac{25}{91}.$$

The sum of these probabilities is unity, as it should be, for some one is ultimately sure to win.

11. Suppose the four cards to be drawn in succession. Any suit whatever may come first and leave an equal chance of the following cards being all different. Hence the probability of the first card being favorable is unity.

To fulfil the required condition the second card must be of a different suit from the first. There are left in the pack 51 cards, of which 12 belong to this suit already drawn, so that any one of the remaining 39 will be favorable.

Therefore the probability of the second card being favorable is $\frac{39}{51}$.

If this happens there will be left 50 cards, 26 of which will belong to the two sets not drawn. The probability that some one of these 26 cards will be drawn is $\frac{26}{50}$.

If one of these 26 cards is drawn there will be left 49 cards, 13 of which belong to the remaining suit. The probability of drawing one of these 13 cards is $\frac{13}{49}$.

Taking the product of all these probabilities, we find it to be

$$\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}.$$

12. When five pieces are taken out of the six which the first purse holds, only one will be left.

Hence there is a probability of $\frac{1}{6}$ that the dollar is left in the first purse, and the probability of $\frac{5}{6}$ that it is transferred to the second. In case the transfer is made, the second purse will contain eleven pieces, one of which will be the dollar. When five pieces are taken out of it, the probability that the dollar is one of these five is $\frac{5}{11}$, and the probability that the dollar is left behind is $\frac{6}{11}$.

We now have the following possible results:

1. That the dollar is left in the first purse to begin with. Probability $= \frac{1}{6}$.

2. That it is transferred to the second purse and left there. Probability $= \frac{5}{6} \cdot \frac{6}{11}$.

3. That it is transferred to the second purse and again transferred back to the first.

Probability $= \frac{5}{6} \cdot \frac{5}{11}$.

In the first and last cases it will be in the first purse

and in the middle case in the second. Summing up the respective probabilities, we have:

Probability of being in the first purse $= \frac{36}{66}$;
" " " second " $= \frac{30}{66}$.

Therefore there is a balance of probability in favor of the coin being in the first purse.

13. It being an equal chance which purse is taken, the probability of each is $\frac{1}{2}$.

If the first purse is chosen, the probability of the eagle will be $\frac{4}{6}$, and of the dollar $\frac{2}{6}$. If the second purse is chosen, the probability of the eagle will be $\frac{4}{10}$, and of the dollar $\frac{6}{10}$.

Multiplying each of these probabilities by the first probability, $\frac{1}{2}$, and summing the probabilities which relate to the dollar and those which relate to the eagle, we find:

Total probability of eagle, $\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{5} = \frac{8}{15}$;
Probability of dollar, $\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{3}{5} = \frac{7}{15}$.

§ 271.

1. Probability is $C^n p^s (1-p)^{n-s}$.

(1) $n = 5,\ p = \frac{1}{3},\ s = 1,\ C_s^n = \frac{5}{1} = 5$;
$\therefore$ probability is $5\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 = \frac{80}{243}$.

(2) $n = 5,\ p = \frac{1}{3},\ s = 2,\ C_s^n = C_2^5 = \dfrac{5\,.\,4}{1\,.\,2} = 10$;
$\therefore$ probability is $10\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{80}{243}$.

(3) $n = 5,\ p = \frac{1}{3},\ s = 3,\ C_s^n = C_3^5 = \dfrac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3} = 10$;
$\therefore$ probability is $10\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{40}{243}$.

(4) $n = 5,\ p = \frac{1}{3},\ s = 4,\ C_s^n = C_4^5 = \dfrac{5\,.\,4\,.\,3\,.\,2}{1\,.\,2\,.\,3\,.\,4} = 5$;
$\therefore$ probability is $5\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) = \frac{10}{243}$.

(5) $n = 5,\ p = \frac{1}{3},\ s = 5,\ C_s^n = C_5^5 = 1$;
$\therefore$ probability is $\left(\frac{1}{3}\right)^5 = \frac{1}{243}$.

2. Probability is $C_s^n p^s (1-p)^{n-s}$.

$$n = 6,\ p = \tfrac{1}{4},\ s = 3,\ C_s^n = C_3^6 = \frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3} = 20;$$

$\therefore$ probability that three and no more will live is

$$20\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^3 = \tfrac{540}{4096}.$$

To find the probability that four will live, make $s = 4$, which gives $\frac{135}{4096}$. For five make $s = 5$, which gives

$\frac{18}{4096}$. That all will live is $\frac{1}{4^6} = \frac{1}{4096}$. Hence the total probability is

$$\frac{540 + 135 + 18 + 1}{4096} = \frac{694}{4096} = \frac{347}{2048}.$$

3. Probability is $C_s^n p \; (1 - p)^{n-s}$.

$$n = 4, p = \tfrac{2}{3}, s = 3, \; C_s^n = \frac{4 \,.\, 3 \,.\, 2}{1 \,.\, 2 \,.\, 3} = 4;$$

$\therefore$ probability is $4 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right) = \frac{32}{81}$ that he will win three games and no more. Probability of winning four games is

$$C_4^4 \left(\tfrac{2}{3}\right)^4 = \left(\tfrac{2}{3}\right)^4 = \tfrac{16}{81};$$

$$\therefore \text{probability required is } \tfrac{32}{81} + \tfrac{16}{81} = \tfrac{48}{81} = \tfrac{16}{27}.$$

§ 274.

1. (*a*) No person can live to 70 unless he first lives to 30. The table of mortality shows that out of 89 042 people aged 30, 38 241 live to the age of 70.

Hence the required probability is $\frac{38\,241}{89\,042} = 0.43$.

In the same way we find the remaining probabilities to be:

(*b*) probability is $\frac{14\,198}{89\,042} = 0.159$;
(*c*) " " $\frac{58\,373}{71\,807} = 0.812$;
(*d*) " " $\frac{38\,241}{58\,373} = 0.655$;
(*e*) " " $\frac{14\,198}{38\,241} = 0.371$;
(*f*) " " $\frac{1566}{14\,198} = 0.11$;
(*g*) " " $\frac{236}{1566} = 0.15$;
(*h*) " " $\frac{6}{236} = 0.025$.

2. To solve the question we find at what age half the people aged 40 are still living.

The table gives 81 326 people living at the age of 40. Half this number is 40 663.

Looking for this number in the table, we find it to correspond nearly to the age of 69. Therefore we may expect that half the people aged 40 will live to 69.

3. Probability that a man aged 30 will live to 70 is, by the table, $\frac{38\,241}{89\,042}$.

Probability that a person aged 30 will live to 60 is $\frac{58\,373}{89\,042}$.

Probability that a person aged 60 will live to 70 is $\frac{38\,241}{58\,373}$.

$$\frac{38\,241}{89\,042} = \frac{58\,373}{89\,042} \times \frac{38\,241}{58\,373} = \frac{38\,241}{58\,373},$$

which was to be proven.

4. Probability that he will live is

$$\frac{47\,057}{49\,057} = 0.95\,767.$$

Hence probability of dying is

$$1 - 0.95\,767 = 0.04\,233.$$

$$\$7000 \times 0.04\,233 = \$296.31$$

is the premium that ought to be paid.

5. Probability that any one will live is

$$\frac{85\,368}{92\,397} = 0.9239.$$

The probability that all ten will live is

$$(0.9239)^{10}.$$

6. The easiest way of proceeding with this question is to ascertain the probability that the planing-mill will not burn down within the limit of time for which it is insured. Subtracting this probability from unity, we shall have the probability of the loss, which, multiplied by the insurance, will give the premium.

The probability that it will not burn down the first year is, by the terms of the question, $\frac{2}{3}$. If it escapes, there will be a probability of $\frac{2}{3}$ that it will survive another year. Therefore, by § 268, Th. II., the probability that it will last two years is $(\frac{2}{3})^2$.

Continuing the reasoning, the probability that it will last n years is $(\frac{2}{3})^n$. We then find insurance for one year $\frac{1}{3}$ of amount $= \$1000$.

For two years $(1 - \frac{4}{9})$ of amount $= \$1666.67$.
For three years $(1 - \frac{8}{27})$ of amount $= \frac{19}{27} = \$2111.11$.
For four years $(1 - \frac{16}{81})$ of amount $= \frac{65}{81} = \$2407.40$.

7. Reasoning as in the last problem, the probability at any epoch that the house will survive one year is $1 - p$. Therefore the probability that it will survive s years is $(1 - p)^s$.

The probability that it will burn within the limit of s years is $1 - (1 - p)^s$; multiplying this by the amount a, we have $\{1 - (1 - p)^s\}\, a$, the premium.

8. In order that a couple may celebrate their golden wedding the man of 25 must live to 75, and the wife of 20 to 70. Reasoning as in Ex. 1, we find the required probability to be

$$\frac{25\,242}{92\,397} \times \frac{38\,241}{93\,014} = 0.1123.$$

9. Since the company must pay the money if either of them die, it can avoid payment only in case they both

live. We readily find the probability that they both will live through the five years to be

$$\frac{25\,942}{35\,241} \times \frac{65\,748}{71\,807} = 0.62\,105.$$

Subtracting this from unity, we find the probability that the company will have to pay to be .37 895. The product of this by \$5000 is \$1894.75, the required premium.

10. The company will not have to pay the money except upon the concurrence of two events, the death of the wife before the age of 55, and the survival of the man to the age of 70. The probability that the wife of 35 will live to 55 is $\frac{65\,748}{85\,368} = 0.7701.$

Subtracting this from unity, the probability that the wife will die is 0.2299. The probability that the husband of 50 will live to the age of 70 is

$$\frac{38\,241}{71\,807} = 0.5325.$$

Multiplying this by the former probability, we find the probability that the company will have to pay the money to be 0.122 421.

Multiplying this by \$10 000, the amount to be paid, the premium is \$1224.21.

§ 276.

1. $\dfrac{1}{(n+2)(n+3)}$.

3. $\dfrac{x^{n-1}}{n\,!}$.

2. $(2n-1)(2n)$.

4. $\dfrac{a^{(2)n-1}}{(n+1)\,z^n}$.

5. $\frac{3}{5}$, $\frac{15}{17}$, $\frac{35}{37}$, $\frac{63}{65}$, etc.

6. $1\,.\,2\,.\,3x + 2\,.\,3\,.\,4x^2 + 3\,.\,4\,.\,5\,x^3 + 4\,.\,5\,.\,6\,x^4 +$ etc.

7. To find 1st term put $n = 0$; 2d, $n = 1$, etc., and we have

$$\frac{3\,.\,4}{5\,.\,6}x + \frac{4\,.\,5}{6\,.\,7}x^2 + \frac{5\,.\,6}{7\,.\,8}x^3 + \frac{6\,.\,7}{8\,.\,9}x^4 + \text{etc.}$$

8. To find 1st term put $n = 2$; 2d, $n = 3$, etc., and we have

$$\frac{3}{1\,.\,2} + \frac{8}{1\,.\,2\,.\,3} + \frac{15}{1\,.\,2\,.\,3\,.\,4} + \frac{24}{1\,.\,2\,.\,3\,.\,4\,.\,5} + \text{etc.}$$

§ 278.

1. $\sum_{j=1}^{j=7} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$
$= 1 + 4 + 9 + 16 + 25 + 36 + 49 = 140.$

2. $\sum_{n=1}^{n=6} n(n-1) = 1.0 + 2.1 + 3.2 + 4.3 + 5.4 + 6.5$
$= 2 + 6 + 12 + 20 + 30 = 70.$

3. $\sum_{n=1}^{n=6} n(n+1) = 1.2 + 2.3 + 3.4 + 4.5 + 5.6 + 6.7$
$= 2 + 6 + 12 + 20 + 30 + 42 = 112.$

4. $\sum_{i=4}^{i=8} m_i = m_4 + m_5 + m_6 + m_7 + m_8.$

5. $\sum_{n=4}^{n=7} nk = 4k + 5k + 6k + 7k = 22k.$

6. $\sum_{n=0}^{n=6} (n+1)(j-1) = (j-1) + 2(j-1) + 3(j-1)$
$+ 4(j-1) + 5(j-1) + 6(j-1) + 7(j-1)$
$= 28(j-1).$

7. $\sum_{i=2}^{i=4} im_i = 2m_2 + 3m_3 + 4m_4.$

8. $\sum_{n=2}^{n=5} n^2m^2 = 2^2m^2 + 3^2m^2 + 4^2m^2 + 5^2m^2 = 54m^2.$

9. $\sum_{n=0}^{n=5} \frac{n-1}{n+1} = -1 + \frac{0}{2} + \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} = \frac{11}{10}.$

10. $\sum_{i=0}^{i=4} h_i.$

12. $\sum_{n=1}^{n=4} n(n+1).$

11. $\sum_{n=1}^{n=4} n^3.$

13. $\sum_{n=1}^{n=5} \frac{n}{n+1}.$

§ 282.

1. $\frac{1}{1-x} = a_0 + a_1x + a_2x^2 + a_3x^4 + a_4x^4 + \text{etc.};$
$\therefore 1 = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3$
$+ (a_4 - a_3)x^4 + \text{etc.}$

Hence
$$\begin{aligned} a_0 &= 1; \\ a_1 - a_0 &= 0, \text{ or } a_1 = a_0 = 1; \\ a_2 - a_1 &= 0, \text{ or } a_2 = a_1 = 1; \\ a_3 - a_2 &= 0, \text{ or } a_3 = a_2 = 1; \\ a_4 - a_3 &= 0, \text{ or } a_4 = a_3 = 1; \\ &\text{etc.} \qquad \text{etc.}; \end{aligned}$$
and we have
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \text{etc.}$$

2. $$\frac{1}{1-2x} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \text{etc.},$$
or $$1 = a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + (a_3 - 2a_2)x^3 + \text{etc.};$$
$$\begin{aligned} \therefore a_0 &= 1; \\ a_1 - 2a_0 &= 0, \text{ or } a_1 = 2a_0 = 2; \\ a_2 - 2a_1 &= 0, \text{ or } a_2 = 2a_1 = 4; \\ a_3 - 2a_2 &= 0, \text{ or } a_3 = 2a_2 = 8; \end{aligned}$$
$$\therefore \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \text{etc.}$$

3. $$\frac{1-x}{1+x} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \text{etc.};$$
$$1 - x = a_0 + (a_1 + a_0)x + (a_2 + a_1)x^2 + (a_3 + a_2)x^3 + (a_4 + a_3)x^4 + \text{etc.}$$
Hence
$$\begin{aligned} a_0 &= 1; \\ a_1 + a_0 &= -1, \text{ or } a_1 = -a_0 - 1 = -2; \\ a_2 + a_1 &= 0, \text{ or } a_2 = -a_1 = 2; \\ a_3 + a_2 &= 0, \text{ or } a_3 = -a_2 = -2; \\ a_4 + a_3 &= 0, \text{ or } a_4 = -a_3 = 2; \\ &\text{etc.} \qquad \text{etc.} \end{aligned}$$
$$\therefore \frac{1-x}{1+x} = 1 - 2x + 2x^2 - 2x^3 + 2x^4 - \text{etc.}$$

4. $$\frac{1+x}{1-x} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \text{etc.};$$
$$1 + x = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3 + (a_4 - a_3)x^4 + \text{etc.}$$
Hence
$$\begin{aligned} a_0 &= 1; \\ a_1 - a_0 &= 1, \text{ or } a_1 = a_0 + 1 = 2; \\ a_2 - a_1 &= 0, \text{ or } a_2 = 2; \\ a_3 - a_2 &= 0, \text{ or } a_3 = 2; \\ &\text{etc.} \qquad \text{etc.} \end{aligned}$$

$$\therefore \frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \text{etc.}$$

5. $$\frac{1+x}{1+2x+3x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \text{etc.};$$

or $$1 + x = a_0 + (a_1 + 2a_0)x + (a_2 + 2a_1 + 3a_0)x^2 + (a_3 + 2a_2 + 3a_1)x^3 + (a_4 + 2a_3 + 3a_2)x^4 + \text{etc.}$$

Hence $a_0 = 1;$

$$\begin{aligned}
a_1 + 2a_0 &= 1, \text{ or } a_1 = 1 - 2a_0 &= -1;\\
a_2 + 2a_1 + 3a_0 &= 0, \text{ or } a_2 = -2a_1 - 3a_0 &= -1;\\
a_3 + 2a_2 + 3a_1 &= 0, \text{ or } a_3 = -2a_2 - 3a_1 &= 5;\\
a_4 + 2a_3 + 3a_2 &= 0, \text{ or } a_4 = -2a_3 - 3a_2 &= -7;\\
a_5 + 2a_4 + 3a_3 &= 0, \text{ or } a_5 = -2a_4 - 3a_3 &= -1;\\
a_6 + 2a_5 + 3a_4 &= 0, \text{ or } a_6 = -2a_5 - 3a_4 &= 23;
\end{aligned}$$

$$\therefore \frac{1+x}{1+2x+3x^2} = 1 - x - x^2 + 5x^3 - 7x^4 - x^5 + 23x^6 + \text{etc.}$$

6. $$\frac{1-x}{1-2x+x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \text{etc.};$$

or $$1 - x = a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1 + a_0)x^2 + (a_3 - 2a_2 + a_1)x^3 + (a_4 - 2a_3 + a_2)x^4 + (a_5 - 2a_4 + a_3)x^5 + \text{etc.}$$

Hence $a_0 = 1;$

$$\begin{aligned}
a_1 - 2a_0 &= -1, \text{ or } a_1 &= 1;\\
a_2 - 2a_1 + a_0 &= 0, \text{ or } a_2 = 2a_1 - a_0 &= 1;\\
a_3 - 2a_2 + a_1 &= 0, \text{ or } a_3 = 2a_2 - a_1 &= 1;\\
a_4 - 2a_3 + a_2 &= 0, \text{ or } a_4 = 2a_3 - a_2 &= 1;
\end{aligned}$$

$$\therefore \frac{1-x}{1-2x+x^2} = 1 + x + x^2 + x^3 + x^4 + x^5 + \text{etc.}$$

7. $$\frac{1-2x+3x^2}{1+2x+3x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \text{etc.};$$

or

$$1 - 2x + 3x^2 = a_0 + (a_1 + 2a_0)x + (a_2 + 2a_1 + 3a_0)x^2 + (a_3 + 2a_2 + 3a_1)x^3 + (a_4 + 2a_3 + 3a_2)x^4 + \text{etc.}$$

Hence $a_0 = 1;$

$$\begin{aligned}
a_1 + 2a_0 &= -2, \text{ or } a_1 = -2 - 2a_0 &= -4;\\
a_2 + 2a_1 + 3a_0 &= 3, \text{ or } a_2 = 3 - 2a_1 - 3a_0 &= 8;
\end{aligned}$$

$$\begin{aligned}
a_3 + 2a_2 + 3a_1 &= 0, \text{ or } a_3 = -2a_2 - 3a_1 = -4;\\
a_4 + 2a_3 + 3a_2 &= 0, \text{ or } a_4 = -2a_3 - 3a_2 = -16;\\
a_5 + 2a_4 - 3a_3 &= 0, \text{ or } a_5 = -2a_4 - 3a_3 = 44;\\
a_6 + 2a_5 - 3a_4 &= 0, \text{ or } a_6 = -2a_5 - 3a_4 = -40;
\end{aligned}$$

$$\therefore \frac{1-2x+3x^2}{1+2x+3x^2} = 1 - 4x + 8x^2 - 4x^3 - 16x^4 + 44x^5 - 40x^6 + \text{etc.}$$

8. $$\frac{1-x}{1+x-x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \text{etc.};$$

$$1 - x = a_0 + (a_1 + a_0)x + (a_2 + a_1)x^2 + (a_3 + a_2 - a_0)x^3 + (a_4 + a_3 - a_1)x^4 + (a_5 + a_4 - a_2)x^5 + \text{etc.}$$

Hence $a_0 = 1;$

$$\begin{aligned}
a_1 + a_0 &= -1, \text{ or } a_1 = -2;\\
a_2 + a_1 &= 0, \text{ or } a_2 = a_1 - \quad = 2;\\
a_3 + a_2 - a_0 &= 0, \text{ or } a_3 = a_0 - a_2 = -1;\\
a_4 + a_3 - a_1 &= 0, \text{ or } a_4 = a_1 - a_3 = -1;\\
a_5 + a_4 - a_2 &= 0, \text{ or } a_5 = a_2 - a_4 = 3;\\
a_6 + a_5 - a_3 &= 0, \text{ or } a_6 = a_3 - a_5 = -4;\\
&\text{etc.} \qquad \text{etc.}
\end{aligned}$$

$$\therefore \frac{1-x}{1+x-x^2} = 1 - 2x + 2x^2 - x^3 - x^4 + 3x^5 - 4x^6 \text{ etc.}$$

§ 283.

1. $$\frac{1-2x}{1+x}.$$

$$\begin{array}{l|l}
1 - 2x & 1 + x \\ \hline
1 + x & 1 - 3x + 3x^2 - 3x^3 + 3x^4 - 3x^5 + 3x^6 \text{ etc.} \\
\quad -3x & \\
\quad -3x - 3x^2 & \\
\qquad\quad 3x^2 & \\
\qquad\quad 3x^2 + 3x^3 & \\
\qquad\qquad -3x^3 & \\
\qquad\qquad -3x^3 - 3x^4 & \\
\qquad\qquad\qquad 3x^4 & \\
\qquad\qquad\qquad 3x^4 + 3x^5 & \\
\qquad\qquad\qquad\quad -3x^5 & \\
\qquad\qquad\qquad\quad -3x^5 - 3x^6 & \\
\qquad\qquad\qquad\qquad\quad 3x^6 \text{ etc.} &
\end{array}$$

2.
$$\frac{1+x}{1-x+x^2}.$$

$$\begin{array}{l|l}
1+x & 1-x+x^2 \\
\underline{1-x+x^2} & \overline{1+2x+x^2-x^3-2x^4-x^5} \\
\quad 2x-x^2 & \\
\quad \underline{2x-2x^2+2x^3} & \\
\qquad x^2-2x^3 & \\
\qquad \underline{x^2-\ \ x^3+x^4} & \\
\qquad\quad -x^3-x^4 & \\
\qquad\quad \underline{-x^3+x^4-x^5} & \\
\qquad\qquad -2x^4+x^5 & \\
\qquad\qquad \underline{-2x^4+2x^5-2x^6} & \\
\qquad\qquad\quad -x^5+2x^6 & \\
\qquad\qquad\quad \underline{-x^5+\ \ x^6-x^7} & \\
\qquad\qquad\qquad x^6-x^7 \text{ etc.} &
\end{array}$$

§ 284.

1. In (b) the coefficients of x and z must vanish;

$$\therefore am + a'n + a'' = 0;$$
$$cm + c'n + c'' = 0.$$

We find

$$m = \frac{c''a' - c'a''}{c'a - ca'};$$
$$n = \frac{ca'' - c''a}{c'a - ca'}.$$

Substituting these values in (b), the coefficients of x and z vanish and we have

$$\left[\frac{b(c''a' - c'a'') + b'(ca'' - c''a)}{c'a - ca'} + b''\right] y$$
$$= \frac{h(c''a' - c'a'') + h'(ca'' - c''a)}{c'a - ca'} + h'';$$

$$\therefore y = \frac{h(c''a' - c'a'') + h'(ca'' - c''a) + h''(c'a - ca')}{b(c''a' - c'a'') + b'(ca'' - c''a) + b''(c'a - ca')}.$$

Similarly,

$$z = \frac{h(a''b' - b''a') + h'(ab'' - a''b) + h''(a'b - ab')}{c(a''b - ab'') + c'(ab'' - a''b) + c''(a'b - ab')}.$$

§ 284 (a).

1. It is evident that the product will contain only odd powers of x, and so will be of the form

$$A_1x + A_3x^3 + A_5x^5 + \text{etc.}$$

We find by performing the multiplication,

$$A_1 = 1;$$
$$A_3 = \frac{1}{3\,!} + \frac{1}{2\,!} = \frac{4}{3\,!};$$
$$A_5 = \frac{1}{5\,!} + \frac{1}{2\,!}\cdot\frac{1}{3\,!} + \frac{1}{4\,!}\cdot\frac{1}{1\,!};$$
$$A_7 = \frac{1}{7\,!} + \frac{1}{2\,!}\cdot\frac{1}{5\,!} + \frac{1}{4\,!}\cdot\frac{1}{3\,!} + \frac{1}{6\,!}\cdot\frac{1}{1\,!};$$

and, in general,

$$A_n = \frac{1}{n\,!} + \frac{1}{2\,!\,(n-2)!} + \frac{1}{4\,!\,(n-4)!} + \dots + \frac{1}{(n-1)!}.$$

Multiplying both sides of this equation by $n!$, we have

$$n!\,A_n = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n-1}$$
$$= C_0^n + C_2^n + C_4^n + \dots + C_{n-1}^n.$$

If we subtract from this sum the alternate combinations $C_1^n + C_3^n + C_5^n + \dots + C_n^n$, the remainder will, by § 262, Th. II., be zero. Therefore the two sums are equal.

But, by § 262, Th. I., the sum of both sums is 2^n. Therefore each sum is half this, whence

$$n\,!\,A_n = \tfrac{1}{2}\,2^n,$$

and $$A_n = \frac{1}{2}\frac{2^n}{n\,!}.$$

Substituting this general value of A_n, and putting $n = 1$, $n = 3$, $n = 5$, etc., we have

$$\text{Product of series} = \frac{1}{2}\left\{2x - \frac{2^3x^3}{3\,!} + \frac{2^5x^5}{5\,!} - \frac{2^7x^7}{7\,!} + \text{etc.}\right\}.$$

2. It is evident that the square of each series will contain only even powers of x. Now if we square the series

$$a_0 + a_2x^2 + a_4x^4 + \text{etc.}, \qquad (1)$$

and suppose the square to be represented by

$$A_0 + A_2x^2 + A_4x^4 + A_6x^6 + \text{etc.}, \qquad (2)$$

we shall have

$$A_n = a_0a_n + a_2a_{n-2} + a_4a_{n-4} + \dots + a_na_0.$$

To put the first series in the form (1), we have

$$a_0 = 1;\ \ a_2 = -\frac{1}{2\,!};\ \dots\ a^n = \pm\frac{1}{n\,!}.$$

Then

$$A_0 = a_0^2 = 1;$$
$$A_2 = a_0a_2 + a_2a_0 = -\,2\,\frac{1}{2\,!};$$

$$A_4 = a_0a_4 + a_2^2 + a_4a_0 = \frac{1}{4\,!} + \frac{1}{2\,!}\cdot\frac{1}{2\,!} + \frac{1}{4\,!};$$

and, in general,

$$A_n = \pm\left(\frac{1}{n!} + \frac{1}{2\,!}\cdot\frac{1}{(n-2)!} + \frac{1}{4\,!}\cdot\frac{1}{(n-4)!} + \ldots + \frac{1}{n\,!}\right),$$

the sign being $+$ when $\frac{n}{2}$ is even, and $-$ when $\frac{n}{2}$ is odd.

By substituting these values in (2), we shall have the square of the first series.

In the second of the given series we have

$$a_1 = 1,\ a_3 = -\frac{1}{3\,!},\ a_5 = \frac{1}{5\,!},\ \ldots\ a_n = \pm\frac{1}{n\,!}.$$

Supposing the square expressed in the form (2), but accenting the A's to distinguish them, we shall have

$$A'_0 = 0;$$

$$A'_2 = a_1a_1 = 1;$$

$$A'_4 = a_1a_3 + a_3a_1 = -\left(\frac{1}{1}\cdot\frac{1}{3\,!} + \frac{1}{3\,!}\cdot\frac{1}{1}\right);$$

$$A'_6 = a_1a_5 + a_3a_3 + a_5a_1$$
$$= \frac{1}{1}\cdot\frac{1}{5!} + \frac{1}{3!}\cdot\frac{1}{3!} + \frac{1}{5!}\cdot\frac{1}{1};$$

and, in general,

$$A'_n = \pm\left(\frac{1}{1}\cdot\frac{1}{(n-1)!} + \frac{1}{3\,!}\cdot\frac{1}{(n-3)!} + \ldots + \frac{1}{n!}\cdot\frac{1}{1}\right),$$

the sign being $+$ when $\frac{n}{2}$ is odd, and $-$ when it is even.

By substituting these values of A in (2), we shall have the square of the second series.

3. To add the two squares together, we must add the coefficients of like powers of x; that is, the corresponding values of A_n. So if we put

$$S_n = A_n + A'_n,$$

we have

$$S_0 = 1;$$

$$S_2 = 1 - \frac{2}{2!} = 0;$$

$$S_4 = \frac{1}{4\,!} - \frac{1}{3\,!}\cdot\frac{1}{1\,!} + \frac{1}{2\,!}\cdot\frac{1}{2\,!} - \frac{1}{1\,!\,3\,!} + \frac{1}{4\,!}$$
$$= \frac{1}{24} - \frac{1}{6} + \frac{1}{4} - \frac{1}{6} + \frac{1}{24} = 0;$$

and, in general, for all values of n,

$$S_n = \pm\left(\frac{1}{n!} - \frac{1}{1\,!\,(n-1)!} + \frac{1}{2\,!\,(n-2)!} - \ldots + \frac{1}{n\,!}\right).$$

It must be observed that the accented A's are always of the opposite sign from those of the first set, which are not accented. Hence if we put the sign $+$ before the second member of the last equation when $\frac{n}{2}$ is even, and the sign $-$ when $\frac{n}{2}$ is odd, we shall have the correct sign for all the terms.

Now multiply both members of the last equation by $n\,!$.

$$n\,!\,S_n =$$
$$\pm\left(1 - \frac{n!}{1!\,(n-1)!} + \frac{n!}{2!\,(n-2)!} - \frac{n!}{3!\,(n-3)!} + \text{etc.}\right).$$

By §§ 257, 259, this equation reduces to

$$n\,!\,S_n = \pm(C_0^n - C_1^n + C_2^n - C_3^n + \ldots) = 0,$$

by § 262, Th. II.

Therefore for all values of n, except $n = 0$, we have

$$S_n = 0,$$

and the sum of the squares of the two series reduces to

$$S_0 = 1.$$

§ 288.

1. Total number of shot is

$$N_1 = \frac{n\,(n+1)}{2}.$$

There are $(n - r)$ rows removed, or

$$\frac{(n-r)\,(n-r+1)}{2} \text{ shot removed;}$$

$$\therefore \text{ No. shot } N = \frac{n\,(n+1) - (n-r)\,(n-r+1)}{2}$$
$$= \frac{r\,(2n-r+1)}{2}.$$

2. $$N_9 = \frac{h\,(h+1)}{2} \text{ total number;}$$

$$N_1 = \frac{s\,(s+1)}{2} \text{ number removed;}$$

$$\therefore N = \frac{h\,(h+1) - s\,(s+1)}{2} = \frac{(h-s)\,(h+s+1)}{2}.$$

3. The successive rows form an arithmetical progression of which the extreme terms are m and n, and the number of terms (rows) $n - m + 1$. Hence

$$N = \frac{(n - m + 1)(n + m)}{2} = \frac{n(n+1) - m(m-1)}{2}.$$

4. In bottom row $= (p + k - 1)$;
In top row $= k$;

$$\therefore N = \frac{p(p + k - 1 + k)}{2} = \frac{p(p + 2k - 1)}{2}.$$

5. Each triangular number being formed by adding to the preceding the numbers 1, 2, 3, 4, etc., we see that if an even number is added, the character of the sum will be unchanged, whereas adding an odd number will change it from even to odd or from odd to even. Hence only the alternate additions will change the character, and the even and odd numbers succeed each other in pairs, thus:

1, 3; 6, 10; 15, 21; 28,36; etc.

The same result may be reached by considering that of the factors $n(n+1)$ one must be even and the other odd.

If half the even factor is even, then $\frac{n(n+1)}{2}$ will be even; but if odd, this product will be odd. Therefore if $\frac{n+1}{2}$ is even, we shall have

$$n\frac{n+1}{2}, \text{ even};$$
$$\frac{n+1}{2}(n+2), \text{ even};$$
$$(n+2)\frac{n+3}{2}, \text{ odd};$$
$$\frac{n+3}{2}(n+4), \text{ odd};$$

etc. etc.

6. The row through the centre of a hexagon will contain $n + (n - 1)$ balls, or $2n - 1$ balls. Taking that as the bottom row, we have a pile of n rows, with $2n - 1$ in its bottom row and n in the top row. In this pile are

$$N_0 = \frac{n(3n - 1)}{2}.$$

Then, on the other side, below the middle row we have a pile of $n - 1$ rows, having n in its bottom row and $2n - 2$ in its top row. The number of balls are

$$N_1 = \frac{(n-1)(3n-2)}{2};$$

$$\therefore N = N_0 + N_1 = \frac{n(3n-1)+(n-1)(3n-2)}{2}$$
$$= 3n^2 - 3n + 1.$$

§ 289.

1. $$N = \frac{n(n+1(n+2)}{1.2.3};$$
$$n = 9;$$
$$\therefore N = \frac{9.10.11}{1.2.3} = 165.$$

2. $$N_0 = \frac{n(n+1)(n+2)}{1.2.3};$$
$$N_1 = \frac{k(k+1)(k+2)}{1.2.3};$$
$$\therefore N = N_0 - N_1 = \frac{n(n+1)(n+2) - k(k+1)(k+2)}{1.2.3}.$$

3. $$N_0 = \frac{n(n+1)(n+2)}{1.2.3};$$
$$N_1 = \frac{(m-1)m(m+1)}{1.2.3};$$
$$\therefore N = N_0 - N_1 = \frac{n(n+1)(n+2) - (m-1)m(m+1)}{1.2.3}.$$

§ 293.

1. $S_0 = n = 20;\ d = 1;\ a_1 = 1;\ a_{21} = 21.$

When $m=1$, $S_1 = \dfrac{21^2 - 1^2}{2} - \dfrac{S_0}{2} = 210.$

When $m=2$, $S_2 = \dfrac{21^3 - 1^3}{3} - S_1 - \dfrac{S_0}{3} = 2870.$

When $m=3$, $S_3 = \dfrac{21^4 - 1^4}{4} - \dfrac{3}{2}S_2 - \dfrac{3.2}{1.2.3}S_1 - \dfrac{3.2.1}{1.2.3.4}S_0$
$= 44\,100.$

2. $$S_1 = n\frac{a_n + a_1}{2};$$
$$\therefore\ S_1 = r\frac{(2r-1)+1}{2} = r^2;$$

$a_1 = 1;$
$a_n = 2r - 1;$
$a_{n+1} = 2r + 1;$
$n = r = S_0.$

$$S_2 = \frac{a_{n+1}^3 - a_1^3}{3.2} - 2\,S_1 - 4\,\frac{S_0}{3}$$
$$= \frac{(2\,r+1)^3 - 1}{3.2} - 2\,r^2 - 4\,\frac{r}{3} = \frac{r(2r-1)\,(2r+1)}{3}.$$

3. $S_0 = r;\ d = 2;\ a_1 = 2;\ a_{n+1} = 2r + 1.$

$$S_1 = r\,\frac{2+2\,r}{2} = r\,(r+1);$$
$$S_2 = \frac{(2\,r+2)^3 - 1}{3.2} - 2\,S_1 - \tfrac{4}{3}\,S_0 = \frac{2\,r\,(r+1)\,(2r+1)}{3}.$$

4. $N_3 = pq + (p-1)\,(q-1) + (p-2)\,(q-2).$

Put $r = p - q;\ p = q + r.$

$$N_3 = (q+r)\,q + (q-1+r)\,(q-1) + (q-2+r)\,(q-2)$$
$$= q^2 + (q-1)^2 + (q-2)^2 + r\,(q + q - 1 + q - 2).$$

To form the sum of the squares we have

$a_1 = q;\ d = -1;\ n = 3;\ S_1 = q + q - 1 + q - 2 = 3(q-1);$
$S_0 = n = 3;\ a_{n+1} = q - 3.$

Therefore, by § 290, Eq. (3),

$$S_2 = \frac{(q-3)^3 - q^3}{-3} + 3\,q - 3 - 1$$
$$= 3\,q^2 - 9\,q + 9 + 3\,q - 4 = 3\,q^2 - 6\,q + 5;$$
$$\therefore N_3 = 3\,q^2 - 6\,q + 5 + 3\,r\,(q-1)$$
$$= 3\,q^2 - 6\,q + 5 + 3\,(p-q)\,(q-1)$$
$$= 3\,pq - 3\,(p+q) + 5.$$

This result could have been obtained more easily by reducing the first value of N_3; but we have gone through the general process to show the mode of proceeding in the general case of s courses.

For s courses we have, using the same notation,

$$N_s = q^2 + (q-1)^2 + (q-2)^2 + \text{etc. to } s \text{ terms}$$
$$+ (p-q)(q + q - 1 + q - 2 + \ldots \text{ to } s \text{ terms}).$$

Here, to form the sum of the squares, we have

$a_1 = q;\ d = -1;\ n = s;\ S_0 = s;$
$a_{n+1} = a_{s+1} = q - s;\ a_n = q - s + 1;$

$$S_1 = s\,\frac{q + q - s + 1}{2} = \frac{s\,(2\,q - s + 1)}{2}.$$

From Eq. (3),

$$S_2 = \frac{(q-s)^3 - q^3}{-3} + \frac{s\,(2\,q - s + 1)}{2} - \frac{s}{3}$$
$$= q^2 s - qs^2 + \frac{s^3}{3} + sq - \frac{s^2}{2} + \frac{s}{2} - \frac{s}{3}$$
$$= q^2 s - qs^2 + qs + \frac{s^3}{3} - \frac{s^2}{2} + \frac{s}{6}.$$

The remaining part of N_s is the product

$$(p-q)\,S_1 = \frac{(p-q)\,s\,(2q-s+1)}{2}$$

$$= -q^2 s + \frac{qs^2}{2} - \frac{qs}{2} + pqs - \frac{ps^2}{2} + \frac{ps}{2}.$$

Adding this to S_2, the result is

$$N_s = pqs - \tfrac{1}{2}q\,(s^2 - s) - \tfrac{1}{2}p\,(s^2 - s) + \frac{s^3}{3} - \frac{s^2}{2} + \frac{s}{6}$$

$$= s\,\{pq + (s-1)\,(-\tfrac{1}{2}p - \tfrac{1}{2}q + \tfrac{1}{3}s - \tfrac{1}{6})\}$$

$$= spq - \frac{s\,(s-1)}{6}\,(3p + 3q - 2s + 1).$$

5. $$\sum_{x=1}^{x=5}(a + bx + cx^2) = \begin{array}{rl} a + b + c & (x = 1) \\ + a + 2b + 4c & (x = 2) \\ + a + 3b + 9c & (x = 3) \\ + a + 4b + 16c & (x = 4) \\ + a + 5b + 25c & (x = 5) \\ \hline = 5a + 15b + 55c. & \end{array}$$

6. We find in the same way

$$\sum_{x=1}^{x=b}(a + bx + cx^2) = ba + (1 + 2 + 3 + \ldots + b)\,b + (1^2 + 2^2 + \ldots + b^2)\,c.$$

$$1 + 2 + 3 + b = \frac{b\,(b+1)}{2};$$

$$1^2 + 2^2 + 3^2 + b^2 = \frac{b\,(b+1)\,(2b+1)}{6}.$$

Hence

$$\Sigma = ab + \frac{b\,(b+1)}{2}\,b + \frac{b\,(b+1)\,(2b+1)}{6}\,c$$

$$= b\left\{a + \frac{(b+1)}{2}\left(b + \frac{2b+1}{3}\,c\right)\right\}.$$

§ 295.

1. $$\frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7}, \text{ etc.}$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \text{etc.}$$

$$= \frac{1}{3} - \frac{1}{n+3}.$$

When n is infinite, the sum becomes $= \frac{1}{3}$.

2. $$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}\cdots\cdot\frac{1}{(2n+1)\,(2n+3)}$$
$$=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\frac{1}{2}\left(\frac{1}{7}-\frac{1}{9}\right)\cdots\cdot$$
$$+\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)+\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$$
$$=\frac{1}{2}\left\{\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\frac{1}{7}\cdots\cdot\frac{1}{2n-1}-\frac{1}{2n+1}\right.$$
$$\left.+\frac{1}{2n+1}-\frac{1}{2n+3}\right\}$$
$$=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right).$$

When n is infinite $=\frac{1}{2.3}=\frac{1}{6}.$

3. $$\frac{2}{3}\left(\frac{1}{2}-\frac{1}{5}\right)+\frac{2}{3}\left(\frac{1}{3}-\frac{1}{6}\right)+\frac{2}{3}\left(\frac{1}{4}-\frac{1}{7}\right)\cdots\frac{2}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right)$$
$$+\frac{2}{3}\left(\frac{1}{n+1}-\frac{1}{n+4}\right);$$

collecting positive and negative terms,

$$\frac{2}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\cdots\cdot\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}\right.$$
$$-\frac{1}{5}-\frac{1}{6}\cdots\cdot-\frac{1}{n-1}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}$$
$$\left.-\frac{1}{n+3}-\frac{1}{n+4}\right)$$
$$=\frac{2}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}\right).$$

When n is infinite, the sum becomes

$$\frac{2}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=\frac{2}{3}\cdot\frac{13}{12}=\frac{13}{18}.$$

4. $$\frac{3}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right)+\frac{3}{2}\left(\frac{1}{3}-\frac{1}{5}\right)\cdots\frac{3}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right).$$

Collecting positive and negative terms,

$$\frac{3}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\cdots+\frac{1}{n}\right.$$
$$\left.-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}\cdots-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$
$$=\frac{3}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right).$$

When n is infinite, the sum becomes

$$\frac{3}{2}\left(1+\frac{1}{2}\right)=\frac{9}{4}.$$

5. $$\left(\frac{1}{a}-\frac{1}{a+1}\right)+\left(\frac{1}{a+1}-\frac{1}{a+2}\right)\dots\left(\frac{1}{a+n}-\frac{1}{a+n+1}\right)$$

$$=\frac{1}{a}-\frac{1}{a+n+1}=\frac{1}{a} \text{ when } n \text{ is infinite.}$$

§ 296.

1 $$S_n = a + 3ar + 5ar^2 + 7ar^3 \dots + (2n-1)ar^{n-1};$$
$$rS_n = ar + 3ar^2 + 5ar^3 \dots (2n-3)ar^{n-1} + (2n-1)ar^n;$$
$$S_n(1-r) = a + 2ar + 2ar^2 + 2ar^3 \dots + 2ar^{n-1} - (2n-1)ar^n$$
$$= a + 2ar(1+r+r^2 \dots r^{n-2}) - (2n-1)ar^n$$
$$= a + 2ar\frac{1-r^{n-1}}{1-r} - (2n-1)ar^n;$$
$$\therefore S_n = \frac{2ar(1-r^{n-1})}{(1-r)^2} + \frac{a[1-(2n-1)r^n]}{1-r}.$$

When n becomes infinite and $x < 1$,

$$S_n = \frac{2ar}{(1-r)^2} + \frac{a}{1-r} = \frac{a(1+2)}{(1-r)^2}.$$

2. $$S_n = 2a + 4ar + 6ar^2 + 8ar^3 \dots + 2nar^{n-1};$$
$$rS_n = 2ar + 4ar^2 + 6ar^3 \dots 2(n-1)ar^{n-1} + 2nar^n;$$
$$S_n(1-r) = 2a(1+r+r^2+r^3+\dots r^{n-1}) - 2nar^n;$$
$$S_n = \frac{2a(1-r^n)}{(1-r)^2} - \frac{2nar^n}{1-r}.$$

When n becomes infinite and $r < 1$,

$$S_n = \frac{2a}{(1-r)^2}.$$

3. $$S_n = (a+b)r + (a+2b)r^2 \dots (a+nb)r^n;$$
$$rS_n = (a+b)r^2 \dots [a+(n-1)b]r^n + (a+nb)r^{n+1};$$
$$S_n(1-r) = (a+b)r + br^2 + br^3 \dots br^n - (a+nb)r^{n+1}$$
$$= ar + br(1+r+r^2+r^3+\dots r^{n-1}) - (a+nb)r^{n+1}$$
$$= ar + \frac{br(1-r^n)}{1-r} - (a+nb)(r^{n+1});$$
$$\therefore S_n = \frac{br(1-r^n)}{(1-r)^2} + \frac{ar-(a+nb)r^{n+1}}{1-r}.$$

When n becomes infinite,

$$S_n = \frac{br}{(1-r)^2} + \frac{ar}{1-r} = \frac{(b+a)r - ar^2}{(1-r)^2}.$$

§ 300.

2.

i	Δ_i	Δ_i'	Δ_i''	Δ_i'''
0	5			
		-20		
1	-15		-30	
		-50		$+9$
2	-65		-21	
		-71		$+9$
3	-136		-12	
		-83		$+9$
4	-219		-3	
		-86		
5	-305			

$$\begin{aligned}\Delta_i &= 5 - 20i - 30\frac{i(i-1)}{2} + 9\frac{i(i-1)(i-2)}{6} \\ &= 5 - 20i + 15(-i^2 + i) + \tfrac{3}{2}(i^3 - 3i^2 + 2i) \\ &= 5 - 2i - \tfrac{39}{2}i^2 + \tfrac{3}{2}i^3.\end{aligned}$$

3.

		Δ'	Δ''
March 1,	$341°\ 5'\ 10''.9$		
		$1°\ 0'\ 9''.6$	
2,	$342°\ 5'\ 20''.5$		$-2''.0$
		$1°\ 0'\ 7''.6$	
3,	$343°\ 5'\ 28''.1$		$-2''.0$
		$1°\ 0'\ 5''.6$	
4,	$344°\ 5'\ 33''.7$		$-2''.0$
		$1°\ 0'\ 3''.6$	
5,	$345°\ 5'\ 37''.3$		$-2''.0$
		$1°\ 0'\ 1''.6$	
6,	$346°\ 5'\ 38''.9$		$-2''.0$
		$0°\ 59'\ 59''.6$	
7,	$347°\ 5'\ 38''.5$		

Since $i = n - 1$ (because the first line, which we call the zero one, is that for which $n = 1$),
$\Delta_0 = 341°\ 5'\ 10''.9$; $\Delta_0' = 1°\ 0'\ 9''.6$; $\Delta_0'' = -2''.0$.
We have, by (d),

$$\text{Long. on March } n = 341°\,5'\,10''.9 + (n-1)(1°\ 0'\ 9''.6) + \frac{(n-1)(n-2)}{2} \times -2''.0.$$

4. The zero date being May 5, we have $n = i + 5$. Then $\Delta_0 = 495$; $\Delta_0' = 50 - 35 = 15$; $\Delta_0'' = -5$; $\Delta_0''' = 0$;

$$i = n - 5.$$

If we put W_n for the quantity of water on the nth day of May—that is, at the end of $n - 5$ days after May 5—

the value of W_n will be the same as that of Δ_{n-5} in equation (d). Putting $i = n - 5$ in this equation, we have

$$W_n = \Delta_{n-5} = \Delta_0 + (n-5)\Delta_0' + \frac{(n-5)(n-6)}{2}\Delta_0''$$
$$= 495 + 15(n-5) - \frac{5(n-5)(n-6)}{2},$$

which expresses the quantity of water on May n. Putting $n = 5$, we have $W_5 = 495$; $n = 6$; $W_6 = 510$; etc

Equating the expression to zero after reducing it, we find $\qquad -\frac{5}{2}n^2 + \frac{85}{2}n + 345 = 0,$
or $\times -\frac{2}{5}$, $\qquad n^2 - 17n - 138 = 0.$

The solution of this quadratic equation gives
$$n = 23 \text{ or } -6.$$

The date $n = 23$ means May 23; $n = -6$ means the date we should find by counting back the days negatively after the manner of algebra. Thus, for $n = 0$ we have May 0 = April 30 (this being the day before May 1), whence the date for $n = -6$ means April 24.

This negative result means that if the supply and consumption of water went on according to the same law before as well as after May 5, the conditions of the problem would all be complied with by starting with an empty cistern on April 24. The city would have put in 50 gallons on that and each succeeding day, and the family would have used -20 gallons on that day; that is, it would have put in 20 gallons more than it took out. April 25 it put in 15; April 26, 10; April 27, 5; April 28, 0. April 29 it took out 5, etc.

§ 301.

1. $\Delta u = (x+1)^3 - x^3 + m(x+1)^2 - mx^2 + n(x+1) - nx$
$\quad = 3x^2 + (2m+3)x + m + n + 1;$
$\Delta^2 u = 3(x+1)^2 - 3x^2 + (2m+3)(x+1) - (2m+3)x$
$\quad = 6x + 2m + 6;$
$\Delta^3 u = 6.$

2. $\Delta u = 8x^3 + 12x^2 + 14x + 5;$
$\Delta^2 u = 24x^2 + 48x + 34;$
$\Delta^3 u = 48x + 72;$
$\Delta^4 u = 48.$

3. $\Delta u = 15x^2 + 35x + 15;$
$\Delta^2 u = 30x + 50;$
$\Delta^3 u = 30.$

§ 303.

1. $$X = \frac{7x-8}{3x-1};$$

$$\frac{7}{3} - \frac{7x-8}{3x-1} < \frac{1}{100};$$

$$\frac{21x - 7 - 21x + 24}{9x-3} = \frac{17}{3x-3} < \frac{1}{100};$$

$$\therefore x > 189\tfrac{2}{9};$$

or

$$\frac{17}{9x-3} < \frac{1}{100\,000};$$

$$\therefore x > 188\,889\tfrac{2}{9}.$$

Again,

$$\frac{17}{9x-3} < \frac{1}{1\,000\,000}.$$

$$\therefore x > 1\,888\,889\tfrac{2}{9}.$$

§ 304.

1. $$\text{Limit } \frac{x-a}{x} \text{ when } x = \infty;$$

$$\frac{x-a}{x} = \frac{\dfrac{x-a}{x}}{\dfrac{x}{x}} = \frac{1 - \dfrac{a}{x}}{1}; \qquad \therefore \text{L.} = 1.$$

2. $$\frac{ax+b}{bx+a} = \frac{a + \dfrac{b}{x}}{b + \dfrac{a}{x}}. \quad \text{When } x \text{ becomes infinite, L.} = \frac{a}{b}.$$

3. $$\frac{mx^2}{px^2 - ax} = \frac{\dfrac{mx^2}{x^2}}{\dfrac{px^2 - ax}{x^2}} = \frac{m}{\dfrac{a}{x}}. \quad \text{When } x = \infty, \text{ L.} = \frac{m}{p}.$$

4. $$\frac{1-x}{1-ax} = \frac{\dfrac{1}{x} - 1}{\dfrac{1}{x} - a}. \quad \text{When } x \text{ approaches } \infty, \text{ L.} = \frac{1}{a}.$$

5. $\dfrac{x^2 - a^2}{x - a}$. Let $x = a + \delta$; then we have

$$\frac{(a+\delta)^2 - a^2}{\delta} = \frac{2\,a\delta + \delta^2}{\delta} = 2\,a + \delta.$$

As x approaches the value a, δ becomes less than any assignable quantity. Hence

$$\text{L.} = 2\,a.$$

6. $\dfrac{a+x}{a-x} = \dfrac{\frac{a}{x}+1}{\frac{a}{x}-1}$ (as x approaches ∞). L. $= -1$.

§ 308.

1. $\sqrt{8} = 3\,(1 - \frac{1}{9})^{\frac{1}{2}}$;

$$(1 - \tfrac{1}{9})^{\frac{1}{2}} = 1 - \frac{1}{2.9} - \frac{1}{8.9^2} - \frac{1}{16.9^3} - \frac{1}{128.9^4} - \text{etc.}$$

1st term..................	1.000 000
2d "	—.055 556
3d "	—.001 543
4th "	—.000 086
5th "	—.000 006
Sum of $(1 - \frac{1}{9})^{\frac{1}{2}}$..........	=.942 809

Whence

$$\sqrt{8} = 3 \times \text{sum} = 2.828\,427;$$
$$\sqrt{8} = \sqrt{4.2} = 2\,\sqrt{2};$$

or

$$\sqrt{2} = \frac{\sqrt{8}}{2}.$$

Whence

$$\sqrt{2} = \frac{2.828\,427}{2} = 1.414\,213.$$

2. $n = \frac{1}{2}$. By (c), § 307,

$$(1-x)^n = 1 - nx + \binom{n}{2} x^2 - \binom{n}{3} x^3 + \binom{n}{4} x^4 \ldots \text{etc.};$$
$$n = \frac{1}{2};$$
$$\binom{n}{2} = \frac{\frac{1}{2}\,(\frac{1}{2} - 1)}{1.2} = -\frac{1}{2}\cdot\frac{1}{4};$$
$$\binom{n}{3} = \frac{\frac{1}{2}\,(\frac{1}{2} - 1)\,(\frac{1}{2} - 2)}{1.2.3} = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6};$$
$$\binom{n}{4} = \binom{n}{3}\cdot\frac{(\frac{1}{2} - 3)}{4} = -\frac{1.1.3.5}{2.4.6.8};$$

etc. etc.

Whence

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1.1}{2.4}x^2 - \frac{1.1.3}{2.4.6}x^3 - \frac{1.1.3.5}{2.4.6.8}x^4 \ldots. - \text{etc.}$$

3. $$(1-x)^{-\frac{1}{2}} = 1 - (-\tfrac{1}{2})x + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}x^2 - \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3}x^3 + \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)}{1.2.3.4}x^4 \ldots\ldots$$

$$= 1 + \tfrac{1}{2}x + \tfrac{1}{2}\cdot\tfrac{3}{4}x^2 + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}x^3 + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{8}x^4 + \ldots. + \frac{1.3.5.7\ldots.(2i-1)}{2.4.6.8\ldots.2i}x^i.$$

4. $$\frac{1}{(1+x)^{\frac{1}{2}}} = (1+x)^{-\frac{1}{2}};$$

$$(1+x)^{-\frac{1}{2}} = 1 + (-\tfrac{1}{2})x + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2}x^2 + \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{2.3}x^3 + \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)}{2.3.4}x^4 + \text{etc.}$$

$$(1+x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x + \tfrac{1}{2}\cdot\tfrac{3}{4}x^2 - \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}x^3 + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{8}x^4 - \ldots\ldots + (-1)\,\frac{1.3.5.7\ldots(2i-1)}{2.4.6.8\ldots 2i}\,x_i.$$

5. $$\left(1+\frac{1}{x}\right)^m = 1 + m\,.\,\frac{1}{x} + \binom{m}{2}\frac{1}{x^2} + \binom{m}{3}\frac{1}{x^3} + \binom{m}{4}\frac{1}{x^4} + \ldots + \binom{m}{i}\frac{1}{x^i}.$$

6. $$(1-x)^{\frac{1}{m}} = 1 - \frac{1}{m}x + \frac{\frac{1}{m}\left(\frac{1}{m}-1\right)}{1.2}x^2 - \frac{\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{1.2.3}x^3 + \frac{\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\left(\frac{1}{m}-3\right)}{1.2.3.4}x^4$$

$$+\ldots+(-1)^i\frac{1}{m}\left(\frac{1}{m}-1\right)\ldots\frac{\left(\frac{1}{m}-i+1\right)}{i!}x$$

Whence

$$(1-x)^{\frac{1}{m}}=1-\frac{1}{m}x+\frac{(1-m)}{1.2\,m^2}x^2-\frac{(1-m)(1-2m)}{1.2.3\,m^3}x^3$$
$$+\frac{(1-m)(1-2m)(1-3m)}{1.2.3.4\,m^4}x^4\ldots.$$
$$+(-1)^i\frac{(1-m)(1-2m)\ldots[1-(i-1)m]}{1.2.3\ldots i\,.\,m^i}x^i$$
$$=1+\frac{-1}{m}x+\frac{(-1)(m-1)}{1.2\,m^2}x^2+\frac{(-1)(m-1)(2m-1)}{1.2.3\,m^3}x^3$$
$$+\text{ etc.}$$

7. $$(1+m)^{\frac{1}{m}}=1+\frac{1}{m}m+\frac{\frac{1}{m}\left(\frac{1}{m}-1\right)}{2}m^2$$
$$+\frac{\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{2.3}m^3$$
$$+\frac{\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\left(\frac{1}{m}-3\right)}{1.2.3.4}m^4+\text{ etc.}$$
$$=1+1+\frac{(1-m)}{2}+\frac{(1-m)(1-2m)}{2.3}$$
$$-\frac{(1-m)(1-2m)(1-3m)}{1.2.3.4}+\text{ etc.}$$

8. $$(a-b)^{-3}=(-1)^3(b-a)^{-3}=-b^{-3}\left(1-\frac{a}{b}\right)^{-3};$$
$$\left(1-\frac{a}{b}\right)^{-3}=1-(-3)\frac{a}{b}+\frac{(-3)(-3-1)}{1.2}\left(\frac{a}{b}\right)^2$$
$$-\frac{(-3)(-3-1)(-3-2)}{2.3}\left(\frac{a}{b}\right)^3$$
$$+\frac{-3(-3-1)(-3-2)(-3-3)}{2.3.4}\left(\frac{a}{b}\right)^4-\ldots.$$
$$=1+3\frac{a}{b}+\frac{3.4}{1.2}\frac{a^2}{b^2}+\frac{3.4.5}{1.2.3}\frac{a^3}{b^3}+\ldots.$$
$$+\ldots.\frac{3.4.5\ldots.(i+2)}{1.2.3\ldots i}\frac{a^i}{b^i}.$$

Whence $(a-b)^{-3}=$

$$-\left\{\frac{1}{b^3}+\frac{3a}{b^4}+\frac{3.4}{1.2}\frac{a^2}{b^5}+\ldots\frac{3.4.5\ \ldots\ (i+2)}{1.2.3\ \ldots\ i}\frac{a^i}{b^{i+3}}\right\}.$$

9. $(1-x)^{-m}=(-1)\ \ x^{-m}\left(1-\frac{1}{x}\right)^{-m};$

$$\left(1-\frac{1}{x}\right)^{-m}=1-\frac{(-m)}{1}\frac{1}{x}+\frac{(-m)\,(-m-1)}{1.2}\frac{1}{x^2}$$
$$-\frac{-m\,(-m-1)\,(-m-2)}{1.2.3}\frac{1}{x^3}$$
$$+\frac{(-m)\,(-m-1)\,(-m-2)\,(-m-3)}{1.2.3.4}\frac{1}{x^4}-\ldots.$$
$$\pm\frac{(-m)\,(-m-1)\ldots(-m-i+1)}{1.2.3.4\ldots i}\frac{1}{x^i}$$
$$=1+m\frac{1}{x}+\frac{m\,(m+1)}{2}\frac{1}{x^2}+\frac{m(m+1)(m+2)}{2.3}\frac{1}{x^3}+\ldots.$$
$$+\frac{m\,(m+1)\,(m+2)\ldots.\,(m+i-1)}{1.2.3.4\ldots.i}\frac{1}{x^i}.$$

Whence $(x)^{-m}\,(1-x)^{-m}=$

$$(-1)^m\left\{\frac{1}{x^m}+\frac{m}{1}\frac{1}{x^{m+1}}+\frac{m\,(m+1)}{1.2}\frac{1}{x^{m+2}}\right.$$
$$+\frac{m\,(m+1)\,(m+2)}{1.2.3}\frac{1}{x^{m+3}}+\ldots.$$
$$\left.+\frac{m\,(m+1)\,(\ldots)\ldots(m+i-1)}{1.2.3\ldots.i}\frac{1}{x^{m+i}}\right\}.$$

10. $\left(1+\frac{1}{m}\right)^m=1+m\frac{1}{m}+\frac{m\,(m-1)}{1.2}\frac{1}{m^2}$

$$+\frac{m\,(m-1)\,(m-2)}{1.2.3}\frac{1}{m^3}+\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}\frac{1}{m^4}$$
$$+\ldots.\frac{m\,(m-1)\,(m-2)\ldots(m-i+1)}{1.2.3.4\ldots.i}\frac{1}{m^i}$$

11. $1610=1728-118=(12)^3-118;$

$$(1610)^{\frac{1}{3}}=[(12)^3-118]^{\frac{1}{3}}=12\left(1-\frac{118}{1728}\right)^{\frac{1}{3}};$$
$$\left(1-\frac{56}{864}\right)^{\frac{1}{3}}=1-\frac{1}{3}\cdot\frac{59}{864}-\frac{2}{2.3^2}\frac{(59)^2}{(864)^2}$$
$$-\frac{2.5}{2.3.3^3}\frac{(59)^3}{(864)^3}-\frac{2.5.8}{2.3.4.3^4}\cdot\frac{(59)^4}{(864)^4}-\text{etc.}$$

Sum the terms and we have

1st term................	=	1.000 000 0
2d "	=	− .022 762 4
3d "	=	− .000 518 1
4th "	=	− .000 019 7
5th "	=	− .000 000 9

$$\text{Sum} = \left(1 - \frac{56}{864}\right)^{\frac{1}{3}} \ldots\ldots = \quad .976\,698\,9$$

$$\sqrt[3]{1610} = 12 \times \text{sum} = 11.720\,387.$$

12. $(\sqrt{a} + \sqrt{b})^n = a^{\frac{n}{2}}\left(1 + \sqrt{\frac{b}{a}}\right)^n;$

$$\left(1 + \sqrt{\frac{b}{a}}\right)^n = 1 - n\sqrt{\frac{b}{a}} + \frac{n(n-1)}{1.2}\frac{b}{a}$$
$$+ \frac{n(n-1)(n-2)}{1.2.3}\left(\frac{b}{a}\right)^{\frac{3}{2}}$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\left(\frac{b}{a}\right)^2$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\left(\frac{b}{a}\right)^{\frac{5}{2}} + \text{etc.};$$

$$\therefore (\sqrt{a} + \sqrt{b})^n = a^{\frac{n}{2}} + na^{\frac{n-1}{2}}b^{\frac{1}{2}} + \frac{n(n-1)}{1.2}a^{\frac{n-2}{2}}b$$
$$+ \frac{n(n-1)(n-2)}{1.2.3}a^{\frac{n-3}{2}}b^{\frac{3}{2}}$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^{\frac{n-4}{2}}b^{\frac{4}{2}} + \text{etc.}$$

13. $\varphi(m) = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \binom{m}{3}x^3 + \binom{m}{4}x^4 + \text{etc.};$

$$\varphi(n) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \text{etc.};$$

$$\varphi(m) \times \varphi(n) = 1 + \left\{\binom{m}{1} + \binom{n}{1}\right\}x$$
$$+ \left\{\binom{m}{2} + \binom{m}{1}\binom{n}{1} + \binom{n}{2}\right\}x^2$$
$$+ \left\{\binom{m}{3} + \binom{m}{2}\binom{n}{2} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}\right\}x^3$$

$$+\left\{\binom{m}{4}+\binom{m}{3}\binom{n}{1}+\binom{m}{2}\binom{n}{2}+\binom{m}{1}\binom{n}{3}+\binom{n}{4}\right\}x^4+\text{etc.}$$

$$\varphi(m+n)=1+\binom{m+n}{1}x+\binom{m+n}{2}x^2+\binom{m+n}{3}x^3+\binom{m+n}{4}x^4+\text{etc.}$$

By § 261, $$\binom{m+n}{p}=\binom{m}{p}+\binom{m}{p-1}\binom{n}{1}+\binom{m}{p-2}\binom{n}{2}+\ldots+\binom{m}{1}\binom{n}{p-1}+\binom{n}{p};$$

$$\therefore\ \binom{m+n}{1}=\binom{m}{1}+\binom{n}{1};$$

$$\binom{m+n}{2}=\binom{m}{2}+\binom{m}{1}\binom{n}{1}+\binom{n}{2};$$

$$\binom{m+n}{3}=\binom{m}{3}+\binom{m}{2}\binom{n}{1}+\binom{m}{1}\binom{n}{2}+\binom{n}{3},\ \text{etc.}$$

Whence $$\varphi(m+n)=1+\left\{\binom{m}{1}+\binom{n}{1}\right\}x+\left\{\binom{m}{2}+\binom{m}{1}\binom{n}{1}+\binom{n}{2}\right\}x^2+\left\{\binom{m}{3}+\binom{m}{2}\binom{n}{1}+\binom{m}{1}\binom{n}{2}+\binom{n}{3}\right\}x^3+\text{etc.};$$

which proves that

$$\varphi(m)\times\varphi(n)=\varphi(m+n).$$

§ 310.

1. By § 309 we have the nth terms

$$(6)\quad C_n\left[x^n+nx^{n-1}y+\binom{n}{2}x^{n-2}y^2+\binom{n}{3}x^{n-3}y^3+\ldots\ \text{etc.}\right]$$

and

$$(7)\quad C_nx^n+C_1C_{n-1}x^{n-1}y+C_2C_{n-2}x^{n-2}y^2+C_3C_{n-3}x^{n-3}y^3+\ldots\ \text{etc.}$$

Substitute in (6) and (7) the value of C_n given in (5):

$$C_n=\frac{1}{n!}C_1^n.$$

Whence

(6) becomes $\frac{1}{n!} C_1^n \left[x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \text{etc.}\right]$;

(7) becomes $\frac{1}{n!} C_1^n x^n + C_1 \frac{1}{(n-1)!} C_1^{n-1}x^{n-1}y$

$$+ \frac{1}{1.2} C_1^2 \frac{1}{(n-2)!} C_1^{n-2}x^{n-2}y^2$$
$$+ \frac{1}{1.2.3} C_1^3 \frac{1}{(n-3)!} C_1^{n-3}x^{n-3}y^3$$
$$+ \frac{1}{4!} C_1^4 \frac{1}{(n-4)!} C_1^{n-4}x^{n-4}y^4 + \text{etc.}$$
$$= \frac{1}{n!} C_1^n \left[x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \binom{n}{4}x^{n-4}y^4 + \text{etc.}\right].$$

Since multiplying the second term by $\frac{n}{n}$ we get

$$C_1^n \frac{n}{n!} x^{n-1}y,$$

and similarly the third term $\times \frac{n(n-1)}{n(n-1)}$ gives

$$C_1^n \frac{n(n-1)}{2!\,.\,n!}, \text{ etc.};$$

$\therefore$ (6) and (7) are identical.

2. $\quad 2.7183^2 = 7.38\,915\,489$
$2.7183^{-1} = .3678$
$2.7183^{-2} = .1353$

By (10)

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} + \text{etc.}$$
$$= 1 + 2 + 2 + 1.3333\tfrac{1}{3} + .6666\tfrac{2}{3} + .26\,666 + .08\,888 + .025\,396 + .006\,349 + .001\,411 + .00\,028 + .00\,005$$
$$= 7.38\,901;$$
$$e^{-1} = 1 - 1 + \frac{1}{1\,.\,2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \text{etc.}$$

$$= 1 - 1 + .50\,000 - .16\,666 + .04\,166 - .00\,833 + .00\,138 - .00\,019 + .00\,002$$
$$= 1.54\,306 - 1.17\,518 = .36\,788;$$
$$e^{-2} = 1 - 2 + \frac{2^2}{2\,!} - \frac{2^3}{3\,!} + \frac{2^4}{4\,!} - \frac{2^5}{5\,!} + \frac{2^6}{6\,!} - \ldots \text{ etc.}$$
$$= 3.76\,217 - 3.6268$$
$$= .1353.$$

3. To show the identity of

$$1 + 1 + x + \frac{(1+x)^2}{2\,!} + \frac{(1+x)^3}{3\,!} + \frac{(1+x)^4}{4\,!} + \ldots \frac{(1+x)^n}{n!}$$

and $$e\left(1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots \frac{x^n}{n\,!}\right).$$

Expanding first expression and collecting terms,

$$1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \frac{1}{5\,!} + \frac{1}{6\,!} + \ldots \frac{1}{n\,!}$$
$$+ \left(1 + \frac{2}{2\,!} + \frac{3}{3\,!} + \frac{4}{4\,!} + \frac{5}{5\,!} + \ldots \frac{n}{n\,!}\right)x$$
$$+ \left(\frac{1}{2\,!} + \frac{3\,.\,2}{2!\,3!} + \frac{4\,.\,3}{2!\,4!} + \frac{5\,.\,4}{2!\,5!} + \frac{6\,.\,5}{2!\,6!} + \ldots \frac{n\,(n-1)}{2!\,n!}\right)x^2$$
$$+ \left(\frac{1}{3\,!} + \frac{4\,.\,3\,.\,2}{3!\,4!} + \frac{5\,.\,4\,.\,3}{3!\,5!} + \ldots \frac{n\,(n-1)\,(n-2)}{3!\,n!}\right)x^3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \left(\frac{1}{i\,!} + \frac{(i+1)\,(i)\ldots 2}{i\,!\,(i+1)\,!} + \frac{(i+2)\,(i+1)\,i\ldots 3}{i!\,(i+2)!} \ldots\right)x^i.$$

The coefficient of x^0 is evidently $= e$.
Coefficient of x in first member reduces to

$$1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \text{etc.} = e;$$

coefficient of x^2 is

$$\frac{1}{2\,!}\left(1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \frac{1}{5\,!} + \text{etc.}\right);$$

coefficient of x^3 is

$$\frac{1}{3\,!}\left(1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \text{etc.}\right);$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

coefficient of x^i is

$$\frac{1}{i\,!}\left(1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \text{etc.}\right);$$

∴ first member reduces to

$$e\left(1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\frac{x^4}{4\,!}+\text{etc.}\right),$$

which is identical with the second member.

§ 312.

1. $\log p^2q = \log p^2 + \log q = 2\log p + \log q.$

2. $\log pq^3 = \log p + 3\log q.$

3. $\log p^2q^5 = 2\log p + 5\log q.$

4. $\log pq^2x^3y^4 = \log p + 2\log q + 3\log x + 4\log y.$

5. $\log\frac{x}{p} = \log x - \log p.$ By Th. VIII.

$\frac{x}{p} = xp^{-1}$, $\log xp^{-1} = \log x - \log p.$ By Ths. VII. and IX. Whence

$$\log\frac{x}{p} = \log(xp^{-1}).$$

6. $\log\frac{xy}{pq} = \log xyp^{-1}q^{-1} = \log x + \log y - \log p - \log q.$

7. $\log\frac{xy^2}{pq^2} = \log x + 2\log y - \log p - 2\log q.$

8. $\log\frac{x^n y^3}{p^m q^3} = n\log x + 3\log y - m\log p - 3\log q.$

9. $\log\sqrt{x} = \log x^{\frac{1}{2}} = \frac{1}{2}\log x.$

10. $\log\sqrt[3]{x}\,\sqrt{y} = \log x^{\frac{1}{3}}y^{\frac{1}{2}} = \frac{1}{3}\log x + \frac{1}{2}\log y.$

11. $\log\sqrt{\frac{p}{q}} = \log p^{\frac{1}{2}}q^{-\frac{1}{2}} = \frac{1}{2}\log p - \frac{1}{2}\log q.$

12. $\log\sqrt{a} = \log a^{\frac{1}{2}} = \frac{1}{2}\log a = \frac{1}{2}.$

13. $\log ax = \log x + \log a = \log x + 1.$

14. $\log\frac{x}{a} = \log x - \log a = \log x - 1.$

15. $\log \frac{x}{a^n} = \log x - n \log a = \log x - n.$

16. $\log \frac{a^n p^m}{x^2 y^3} = n + m \log p - 2 \log x - 3 \log y.$

17. $\log \sqrt{a^2 - x^2} = \log (a + x)^{\frac{1}{2}} (a - x)^{\frac{1}{2}} = \frac{1}{2} \log (a + x) + \frac{1}{2} \log (a - x).$

18. $\log \sqrt{1 - x^2} = \log (1 + x)^{\frac{1}{2}} (1 - x)^{\frac{1}{2}} = \frac{1}{2} \log (1 + x) + \frac{1}{2} \log (1 - x).$

19. $\log (a^2 - x^2) = \log (a + x)(a - x) = \log (a + x) + \log (a - x).$

§ 316.

1. Let $b^l = a$ (1),

and $a^{l'} = b;$

$\therefore a^{ll'} = b^l;$

comparing with (1),

$a^{ll'} = a;$

$\therefore ll' = 1$, or $l = \frac{1}{l'}.$

2. It expresses that the natural logarithm of 10 is the reciprocal of the common logarithm of e.

§ 320.

1. $\log 4 = \log 2^2 = 2 \log 2 = 0.60\,206;$

$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - 0.30\,103 = 0.69\,897;$

$\log 6 = \log 3.2 = \log 3 + \log 2 = 0.77\,815;$

$\log 8 = \log 2^3 = 3 \log 2 = 0.90\,309;$

$\log 9 = \log 3^2 = 2 \log 3 = 0.95\,424;$

$\log 12 = \log 3.2^2 = \log 3 + 2 \log 2 = 1.07\,918;$

$\log 12\frac{1}{2} = \log \frac{100}{8} = \log 100 - \log 8 = 1.09\,691$

$\log 15 = \log 3.\frac{10}{2} = \log 3 + \log 10 - \log 2 = 1.17\,609;$

$\log 16 = \log 2^4 = 4 \log 2 = 1.20\,412;$

$\log 16\frac{2}{3} = \log \frac{100}{6} = \log 100 - \log 3 - \log 2 = 1.22\,185;$

$\log 18 = \log 2.3^2 = \log 2 + 2 \log 3 = 1.25\,527;$

$\log 20 = \log 2.10 = \log 2 + \log 10 = 1.30\,103;$

$\log 250 = \log (\frac{10}{2})^3.10 = 3 \log 10 - 2 \log 2 = 2.39\,794;$

$\log 6250 = \log (\frac{10}{2})^4.10 = 5 \log 10 - 4 \log 2 = 3.79\,588$

2. 2^{100}, $\log 2^{100} = 100 \log 2 = 30.10\,300.$ Ans. 31.

3. $$\sqrt{49} = 7;$$
$$\therefore \log 7 = \tfrac{1}{2} \log 49 = .845\,098.$$

4. $$11 = \sqrt[3]{1331};$$
$$\therefore \log 11 = \tfrac{1}{3} \log 1331 = 1.041\,393.$$

5. $$105 = 3 \cdot 7 \cdot 5 = 3\sqrt{49} \cdot \tfrac{10}{2};$$
$$\therefore \log 105 = \log 3 + \tfrac{1}{2}\log 49 + \log 10 - \log 2$$
$$= 2.021\,188;$$
$$1.05 = \tfrac{105}{100};$$
$$\log 1.05 = \log 105 - \log 100 = .021\,188.$$

6. $$\log 1.05^{10} = 10 \log 1.05 = .21\,188.$$

7. $$(1.05)^{1000};$$
$$\log (1.05)^{1000} = 1000 \log 1.05 = 21.188\,000;$$
$$21 + 1 = 22. \quad \text{Ans.}$$

8. By § 319, com. log $(n+1)$ = com. log n +
$$2M\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \text{etc.}\right\}.$$
$$\text{Let } n + 1 = x^2;$$
$$\therefore n = x^2 - 1;$$
$$2n + 1 = 2x^2 - 1.$$
Whence $\log x^2 = \log (x^2 - 1) +$
$$2M\left\{\frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)^3} + \frac{1}{5(2x^2-1)^5} + \text{etc.}\right\},$$
or $2 \log x = \log (x+1) + \log (x-1) +$
$$2M\left\{ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \right\};$$
$\log x = \tfrac{1}{2}\log (x+1) + \tfrac{1}{2}\log (x-1) +$
$$M\left\{\frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)^3} + \frac{1}{5(2x^2-1)^5} + \text{etc.}\right\}.$$

11. $c^{ax} = n, \; ax \log c = \log n;$
$$\therefore x = \frac{\log n}{a \log c}.$$

12. $c^{bx} = \dfrac{1}{m}, \; bx \log c = -\log m;$
$$\therefore x = -\frac{\log m}{b \log c}.$$

13. $b^x = \dfrac{1}{p}, \; x \log b = -\log p;$
$$\therefore x = -\frac{\log p}{\log b}.$$

14. $b^{-x} = p,\quad -x \log b = \log p;$

$$\therefore x = -\frac{\log p}{\log b}, \text{ identical to (13); since}$$

$$b^x = \frac{1}{p}, \text{ or } \frac{1}{b^x} = b^{-x} = p,$$

which is identical with (14).

15. $a^{cx} = m,\quad cx \log a = \log m;$

$$\therefore x = \frac{\log m}{c \log a}.$$

16. $bc^x = k,\quad \log b + x \log c = \log k;$

$$\therefore x = \frac{\log k - \log b}{\log c}.$$

17. (1) $a^x b^y = p$, and (2) $a^y b^x = q$;

$$(1)\ x \log a + y \log b = \log p;$$
$$(2)\ y \log a + x \log b = \log q.$$

Multiplying (1) by $\log a$, and (2) by $\log b$,

$$x \log^2 a + y \log a \log b = \log a \log p$$
$$x \log^2 b + y \log a \log b = \log b \log q$$

$$\therefore x = \frac{\log a \log p - \log b \log q}{\log^2 a - \log^2 b};$$

and similarly

$$y = \frac{\log b \log p - \log a \log q}{\log^2 b - \log^2 a}.$$

§ 324.

1.

$$\begin{array}{r} 3x + 4yi + m \\ 5ni + 2m \\ -6yi + 6m \\ \hline 3x + (5n - 2y)i + 9m \end{array}$$

2.

$$\begin{array}{r} 4ai \\ 17i \\ 3a + 6bi \\ x + \quad yi \\ \hline 3a + x + (4a + 17 + 6b + y)i \end{array}$$

3.

$$\begin{array}{lll} a + bi + m & & \\ \quad -ni & -p & \\ \quad +qi & & \\ \quad -yi & & +z \\ \quad +ui & & \\ \hline a + (b - n + q - y + u)i + m - p + z & & \end{array}$$

4. $a + bi - (m - ni) - (x + yi) =$
$(a - m - x) + (b + n - y)\,i.$

5. $m\,(a - bi) - n\,(x - yi) = (ma - nx) + (ny - mb)\,i.$

§ 325.

1. $(x + yi)\,(a - b) = (a - b)\,x + (ay - by)\,i.$

2. $(m + ni)\,ai = -\,an + ami.$

3. $(m - ni)\,bi = bn + mbi.$

4. $(1 + i)\,(1 - i) = 1 - i^2 = 1 + 1 = 2.$

5. $(x - yi)\,(a + bi) = (ax + by) + (bx - ay)\,i.$

6. $(x - yi)\,(x + yi) = x^2 - y^2i^2 = x^2 + y^2.$

7. $(a - ai - bi)\,(a + ai + bi)$
$= [a - (a + b)\,i]\,[a + (a + b)\,i] = a^2 - (a + b)^2\,i^2$
$= a^2 + (a + b)^2 = 2a^2 + 2\,ab + b^2.$

8. $(a + bi)^2 = a^2 + 2\,abi + b^2i^2 = (a^2 - b^2) + 2\,abi.$

9. $(m + ni)^3 = m^3 + 3\,m^2ni + 3\,m\,(ni)^2 + (ni)^3$
$= (m^3 - 3\,mn^2) + (3\,m^2n - n^3)\,i.$

10. $(1 + i)^2 = 1 + 2\,i + i^2 = 2\,i.$

11. $(1 - i)^2 = 1 - 2\,i + i^2 = -\,2\,i.$

§ 326.

1. $x^2 + 4 = (x + 2\,i)\,(x - 2\,i).$

2. $x^2 + 2 = (x + \sqrt{2}\,i)\,(x - \sqrt{2}\,i).$

3. $x^2 - 2\,x + 5 = (x - 1)^2 + 4$
$= [(x - 1) + 2i]\,[(x - 1) - 2\,i].$

4. $x^2 - 4\,x + 13 = (x - 2)^2 + 9$
$= [(x - 2) + 3\,i]\,[(x - 2) - 3\,i].$

5. $a + b = (\sqrt{a} + \sqrt{b}\,i)\,(\sqrt{a} - \sqrt{b}\,i).$

6. $a^2 + 2\,an + 5\,n^2 = (a + n)^2 + 4\,n^2$
$= [(a + n) + 2\,ni]\,[(a + n) - 2\,ni].$

7. $x^2 + 2\,xy + 2\,y^2 = (x + y)^2 + y^2$
$= [(x + y) + yi]\,[(x + y) - yi].$

§ 328.

1. $7 - 3i - 6i^2 + 2i^3 + i^4 - i^5 =$
$$(7 + 6 + 1) - (3 + 2 + 1)i = 14 - 6i.$$

2. $1 + i - i^2 + i^3 - i^4 - i^5 + i^6 =$
$$1 + i + 1 - i - 1 - i - 1 = -i.$$

3.
$$\frac{2}{i-1} = x + yi;$$
$$2 = -x - y + (x - y)i;$$
$$\therefore -x - y = 2$$
and $\quad x - y = 0$

add, $\quad -2y = 2; \quad \therefore y = -1;$

subtract, $\quad -2x = 2; \quad \therefore x = -1;$

whence $\quad \dfrac{2}{i-1} = -1 - i.$

4. $\dfrac{6 + 5i}{6 - 5i} = x + yi;$
$$6 + 5i = 6x + 5y - (5x - 6y)i.$$
$$\therefore \begin{aligned} 6x + 5y &= 6 \\ 5x - 6y &= -5 \end{aligned} \quad \text{or} \quad \begin{aligned} 30x + 25y &= 30 \\ 30x - 36y &= -30 \\ \hline 61y &= 60 \\ y &= \frac{60}{61} \end{aligned}$$
$$\begin{aligned} 36x + 30y &= 36 \\ 25x - 30y &= -25 \\ \hline 61x &= 11 \\ x &= \frac{11}{61}; \end{aligned}$$
whence $\quad \dfrac{6 + 5i}{6 - 5i} = \dfrac{11}{61} + \dfrac{60}{61}i.$

5.
$$\frac{1+i}{1-i} = x + yi;$$
$$1 + i = x + y - (x - y)i;$$
$$\begin{aligned} x + y &= 1 \\ x - y &= -1 \\ \hline 2x &= 0 \\ x &= 0 \end{aligned} \qquad \begin{aligned} x + y &= 1 \\ x - y &= -1 \\ \hline 2y &= 2 \\ y &= 1; \end{aligned}$$
$$\therefore \frac{1+i}{1-i} = i.$$

6. $$\frac{mi\,(x-ai)}{x+ai} = u + vi;$$

$$\therefore am + mxi = xu - av + (au + xv)\,i.$$

$$\begin{array}{ll}
xu - av = am & \\
\underline{au + xv = mx} & x^2u - axv = \quad axm \\
axu - a^2v = a^2m & \underline{a^2u + axv = \quad axm} \\
\underline{axu + x^2v = x^2m} & (x^2 + a^2)\,u = 2\,axm \\
(x^2+a^2)\,v = (x^2-a^2)\,m & \\
v = \dfrac{(x^2-a^2)\,m}{x^2+a^2} & u = \dfrac{2\,axm}{x^2+a^2};
\end{array}$$

whence

$$\frac{mi\,(x-ai)}{x+ai} = \frac{2\,axm}{x^2+a^2} + \frac{(x^2-a^2)\,m}{x^2+a^2}\,i.$$

7. $$\frac{1-i}{2+4i} = x + yi;$$

$$1 - i = 2x - 4y + (4x + 2y)\,i;$$

$$\begin{array}{ll}
2x - 4y = \quad 1 & \\
\underline{4x + 2y = -1} & \\
4x - 8y = \quad 2 & \\
\underline{4x + 2y = -1} & \\
-10\,y = 3, & y = \dfrac{-3}{10}; \\
2x - 4y = \quad 1 & \\
\underline{8x + 4y = -2} & \\
10\,x = -1, & x = -\dfrac{1}{10};
\end{array}$$

whence $\dfrac{1-i}{2+4i} = -\dfrac{1}{10} - \dfrac{3}{10}\,i.$

8. $$\frac{a+bi}{a-bi} = x + yi;$$

$$a + bi = ax + by - (bx - ay)\,i;$$

$$\begin{array}{ll}
ax + by = \quad a & a^2x + aby = \quad a^2 \\
\underline{bx - ay = -b} & \underline{b^2x - aby = -b^2} \\
y = \dfrac{2\,ab}{a^2+b^2}, & x = \dfrac{a^2-b^2}{a^2+b^2};
\end{array}$$

whence $\dfrac{a+bi}{a-bi} = \dfrac{a^2-b^2}{a^2+b^2} + \dfrac{2\,ab}{a^2+b^2}\,i.$

9. $$\frac{(a+bi)\,(a-bi)}{(x+bi)^2} = \frac{a^2+b^2}{(x+bi)^2} = \frac{a^2+b^2}{(x^2-b^2)+2\,bxi};$$

$$\frac{a^2+b^2}{(x^2-b^2)+2\,bxi}=y+zi;$$

$\therefore a^2+b^2=(x^2-b^2)\,y-2\,bxz+[(x^2-b^2)\,z+2\,bxy]\,i.$

(1) $(x^2-b^2)\,y-2\,bxz=a^2+b^2;$

(2) $(x^2-b^2)\,z+2\,bxy=0$, or $y=-\dfrac{(x^2-b^2)}{2\,bx}\,z.$

Substituting y in (1),

$$-\frac{(x^2-b^2)^2\,z}{2\,bx}-2\,bxz=a^2+b^2;$$

$$\therefore z=-\frac{(a^2+b^2)\,2\,bx}{(x^2-b^2)^2+4\,b^2x^2}=-\frac{(a^2+b^2)\,2\,bx}{(x^2+b^2)^2},$$

and $y=\dfrac{(a^2+b^2)\,(x^2-b^2)}{(x^2+b^2)^2};$

whence

$$\frac{(a+bi)\,(a-bi)}{(x+bi)^2}=\frac{(a^2+b^2)\,(x^2-b^2)}{(x^2+b^2)^2}-\frac{(a^2+b^2)\,2\,bx}{(x^2+b^2)^2}\,i.$$

10. Equation 10, § 310, is

$$e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\frac{x^5}{1.2.3.4.5}+\text{etc.};$$

$x=i;$

$$\therefore e^i=1+i+\frac{i^2}{1.2}+\frac{i^3}{1.2.3}+\frac{i^4}{1.2.3.4}+\text{etc.}$$

$$=1+i-\frac{1}{1.2}-\frac{i}{1.2.3}+\frac{1}{1.2.3.4}+\frac{i}{1.2.3.4.5}-\frac{1}{1.2.3.4.5.6}-\text{etc.}$$

$$=\left(1-\frac{1}{1.2}+\frac{1}{1.2.3.4}-\frac{1}{1.2.3.4.5.6}+\text{etc.}\right)+\left\{1-\frac{1}{1.2.3}+\frac{1}{1.2.3.4.5}-\text{etc.}\right\}i.$$

11. $(1+xi)^n=1+\binom{n}{1}xi+\binom{n}{2}(xi)^2+\binom{n}{3}(xi)^3+\binom{n}{4}(xi)^4+\text{etc.}$

$$=1-\binom{n}{2}x^2+\binom{n}{4}x^4-\binom{n}{6}x^6+\text{etc.}+\left\{\binom{n}{1}x-\binom{n}{3}x^3+\binom{n}{5}x^5=\text{etc.}\right\}i.$$

12. $(1+bi)^n = 1 - \binom{n}{2}b^2 + \binom{n}{4}b^4 - \binom{n}{6}b^6 + \text{etc.}$
$$+ \left\{\binom{n}{1}b - \binom{n}{3}b^3 + \binom{n}{6}b^5 - \text{etc.}\right\} i;$$
$$(1-bi)^n = 1 - \binom{n}{2}b^2 + \binom{n}{4}b^4 - \binom{n}{6}b^6 + \text{etc.}$$
$$- \left\{\binom{n}{1}b - \binom{n}{3}b^3 + \binom{n}{6}b^5 - \text{etc.}\right\} i;$$
whence $(1+bi)^n + (1-bi)^n$
$$= 2\left\{1 - \binom{n}{2}b^2 + \binom{n}{4}b^4 - \binom{n}{6}b^6 + \text{etc.}\right\},$$
and $(1+bi)^n - (1-bi)^n$
$$= 2\left\{\binom{n}{1}b - \binom{n}{3}b^3 + \binom{n}{5}b^5 + \text{etc.}\right\} i.$$

13. $a + ai + ai^2 + ai^3 + ai^4 + ai^5 + ai^6 + ai^7$
$$= a + ai - a - ai + a + ai - a - ai.$$

14. 1st term, a;
nth " ar^{n-1}, $r = \frac{i}{2}$;
$$S = \frac{a - ar^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Since $r = \frac{i}{2}$, a proper fraction, and $1 - r$, a finite quantity; $a\frac{r^n}{1-r}$, as n approaches infinity, becomes less than any assignable quantity.
$$\therefore S = \frac{a}{1-r} = \frac{a}{1-\frac{i}{2}} = \frac{2a}{2-i} = \frac{4}{5}a + \frac{2}{5}ai.$$
We might also proceed by forming the odd and even terms of the development separately, obtaining
$$S = a\left(1 - \tfrac{1}{4} + \tfrac{1}{16} - \tfrac{1}{64} + \ldots\right);$$
$$+ a\left(\tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{32} - \text{etc.}\right) i.$$
Each of these coefficients of a is a geometrical progression of which the common ratio is $-\frac{1}{4}$. The limit of the sum of the first is $\frac{4}{5}$, and of the second $\frac{2}{5}$, which lead to the same result as before.

§ 329.

1. $\sqrt{3+4i} = x + yi;$

$$x^2 - y^2 + 2xyi = 3 + 4i;$$
$$x^2 - y^2 = 3;$$
$$2xy = 4;$$
$$\therefore x = \frac{\sqrt{(\sqrt{3^2+4^2}+3)}}{\sqrt{2}} = 2;$$
$$y = \frac{\sqrt{(\sqrt{3^2+4^2}-3)}}{\sqrt{2}} = 1;$$

whence $\sqrt{3+4i} = 2 + i.$

2. $\sqrt{4+3i} = x + yi;$

$$x = \frac{\sqrt{(\sqrt{4^2+3^2}+4)}}{\sqrt{2}} = \frac{3}{\sqrt{2}};$$
$$y = \frac{\sqrt{(\sqrt{4^2+3^2}-4)}}{\sqrt{2}} = \frac{1}{\sqrt{2}};$$

whence $\sqrt{4+3i} = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$

3. $\sqrt{12+5i} = x + yi;$

$$x = \frac{\sqrt{(\sqrt{12^2+5^2}+12)}}{\sqrt{2}} = \frac{5}{\sqrt{2}};$$
$$y = \frac{\sqrt{(\sqrt{12^2+5^2}-12)}}{\sqrt{2}} = \frac{1}{\sqrt{2}};$$

whence $\sqrt{12+5i} = \frac{5}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$

4. $\sqrt{i} = \frac{\sqrt{(\sqrt{a^2+b^2}+a)}}{\sqrt{2}} + \frac{\sqrt{(\sqrt{a^2+b^2}-a)}}{\sqrt{2}}i.$

Here $a = 0,\ b = 1;$

$$\therefore \sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \quad \text{or} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i;$$

squaring, $\frac{1}{2} + \frac{2}{2}i - \frac{1}{2} = i.$

For $-i$ we have $a=0,\ b=-1$;

$$\therefore\ \sqrt{-i}=\sqrt{\frac{\sqrt{(-1)^2}}{\sqrt{2}}}+\sqrt{\frac{\sqrt{(-1)^2}}{\sqrt{2}}}\,i;$$

$$\sqrt{-i}=\frac{\sqrt{-1}}{\sqrt{2}}+\frac{\sqrt{-1}}{\sqrt{2}}\,i$$

$$=\frac{i}{\sqrt{2}}-\frac{1}{\sqrt{2}}\ \text{ or }\ \frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}};$$

squaring, $\quad -\frac{1}{2}-\frac{2}{2}i+\frac{1}{2}=-i.$

5. $$\sqrt{\sqrt{i}}=\sqrt{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i}=x+yi;$$

$$x=\frac{\sqrt{\left(\sqrt{\frac{1}{2}+\frac{1}{2}}+\frac{1}{\sqrt{2}}\right)}}{\sqrt{2}}=\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}};$$

$$y=\frac{\sqrt{\left(\sqrt{\frac{1}{2}+\frac{1}{2}}-\frac{1}{\sqrt{2}}\right)}}{\sqrt{2}}=\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}};$$

whence $$\sqrt[4]{i}=\frac{\sqrt{\sqrt{2}+1}}{\sqrt{2}\,\sqrt[4]{2}}+\frac{\sqrt{\sqrt{2}-1}}{\sqrt{2}\,\sqrt[4]{2}}\,i.$$

By changing the sign of i we have

$$\sqrt[4]{-i}=\frac{\sqrt{\sqrt{2}+1}}{\sqrt{2}\,\sqrt[4]{2}}-\frac{\sqrt{\sqrt{2}-1}}{\sqrt{2}\,\sqrt[4]{2}}\,i.$$

§ 330.

1. $$x^2+x+1=0;$$
$$x^2+x+\tfrac{1}{4}=-\tfrac{3}{4};$$
$$x+\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{-3}$$
$$=\pm\frac{\sqrt{3}}{2}\,i;$$
$$\therefore x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\,i.$$

2. $$x^2-x+1=0;$$
$$x^2-x+\tfrac{1}{4}=-\tfrac{3}{4};$$
$$x-\tfrac{1}{2}=\pm\frac{\sqrt{-3}}{2}$$
$$=\pm\frac{\sqrt{3}}{2}\,i;$$
$$\therefore x=\frac{1}{2}\pm\frac{\sqrt{3}}{2}.$$

3. $x^2 + 3x + 10 = 0;$
$x^2 + 3x + \frac{9}{4} = -\frac{31}{4};$
$\therefore x + \frac{3}{2} = \pm \frac{\sqrt{-31}}{2}$
$= \pm \frac{\sqrt{31}}{2} i;$
whence
$x = -\frac{3}{2} \pm \frac{\sqrt{31}}{2} i.$

4. $x^2 + 10x + 34 = 0;$
$x^2 + 10x + 25 = -9;$
$\therefore x + 5 = \pm \sqrt{-9}$
$= \pm 3i;$
whence $x = -5 \pm 3i.$

5. $[x - (a + bi)][x - (a - bi)] = 0;$
$\therefore x^2 - 2ax + a^2 + b^2 = 0.$

6. $[x - (ai + b)][x - (ai - b)] = 0;$
$\therefore x^2 - 2axi - a^2 - b^2 = 0.$

§ 332.

1. $(\cos u + i \sin u)^3 = \cos 3u + i \sin 3u;$
$\cos^3 u + 3i \cos^2 u \sin u - 3 \cos u \sin^2 u - i \sin^3 u$
$= \cos 3u + i \sin 3u.$
Equating coefficients, we have
$$\cos 3u = \cos^3 u - 3 \cos u \sin^2 u \quad (1)$$
and $$\sin 3u = -\sin^3 u + 3 \cos^2 u \sin u. \quad (2)$$
Next,
$(\cos u + i \sin u)^4 = \cos 4u + i \sin 4u;$
$\therefore \cos^4 u + 4i \cos^3 u \sin u - 6 \cos^2 u \sin^2 u$
$- 4i \cos u \sin^3 u + \sin^4 u$
$= \cos 4u + i \sin 4u.$
Equating coefficients, we have
$$\cos 4u = \cos^4 u - 6 \cos^2 u \sin^2 u + \sin^4 u \quad (1)$$
and $$\sin 4u = 4 \cos^3 u \sin u - 4 \cos u \sin^3 u. \quad (2)$$

2. Equation (a) is
$$e^{ui} = \cos u + i \sin u.$$
By writing in this equation, successively,
$$u = a; \text{ then } u = b, \text{ then } u = a + b,$$
we have the three equations
$$e^{ai} = \cos a + i \sin a, \quad (1)$$
$$e^{bi} = \cos b + i \sin b, \quad (2)$$
$$e^{(a+b)i} = \cos (a + b) + i \sin (a + b). \quad (3)$$
(1) × (2) gives
$e^{(a+b)i} = \cos a \cos b + i \cos a \sin b + i \cos b \sin a$
$- \sin a \sin b$
$= \cos (a + b) + i \sin (a + b)$, from (3).
Equating coefficients, we have
$$\cos (a + b) = \cos a \cos b - \sin a \sin b$$
and $$\sin (a + b) = \sin a \cos b + \cos a \sin b.$$

3. $(x - i)(x - 2i)(x - 3i)(x - 4i) =$
$$(x^4 - 35x^2 + 24) - (10x^3 - 50x)\,i.$$

4. $$(a + bi)^{\frac{1}{2}} = (a)^{\frac{1}{2}} + \tfrac{1}{2} a^{\frac{1}{2} - 1}\, bi + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!} a^{\frac{1}{2} - 2}(bi)^2$$
$$+ \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!} a^{\frac{1}{2} - 3}(bi)^3 + \dots$$
$$+ \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} a^{\frac{1}{2} - r}(bi)^r;$$

$$\therefore (a + bi)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{bi}{2\,.\,a^{\frac{1}{2}}} + \frac{b^2}{2^2\,.\,2!\,a^{\frac{3}{2}}} - \frac{3\,b^3 i}{2^3\,.\,3!\,a^{\frac{5}{2}}}$$
$$- \frac{3\,.\,5\,.\,b^4}{2^4\,.\,4!\,a^{\frac{7}{2}}} + \frac{3\,.\,5\,.\,7}{2^5\,.\,5!}\frac{b^5}{a^{\frac{9}{2}}}\,i$$
$$+ \frac{3\,.\,5\,.\,7\,.\,9}{2^6\,.\,6!}\frac{b^6}{a^{\frac{11}{2}}} - \frac{3\,.\,5\,.\,7\,.\,9\,.\,11}{2^7\,.\,7!}\frac{b^7}{a^{\frac{13}{2}}}\,i$$
$$+ \dots\dots\dots\dots\dots\dots\dots$$
$$+ \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}\frac{b^r}{a^{r-\frac{1}{2}}}(i)^r$$
$$= a^{\frac{1}{2}}\left(1 + \frac{1}{2^2\,.\,2!}\frac{b^2}{a^2} - \frac{3\,.\,5}{2^4\,.\,4!}\frac{b^4}{a^4} + \frac{3\,.\,5\,.\,7\,.\,9}{2^6\,.\,6!}\frac{b^6}{a^6} - \dots\right.$$
$$\left. - (-1)^p \frac{3\,.\,5\,.\,7\dots(4p-3)}{2^{2p}\,.\,2p!}\frac{b^{2p}}{a^{2p}}\right)$$
$$+ a^{\frac{1}{2}}\left(\frac{1}{2}\frac{b}{a} - \frac{3}{2^3\,.\,3!}\frac{b^3}{a^3} + \frac{3\,.\,5\,.\,7}{2^5\,.\,5!}\frac{b^5}{a^5} - \dots\right.$$
$$\left. + (-1)^s \frac{3\,.\,5\,.\,7\dots[2(2s+1)-3]}{2^{2s+1}(2s+1)!}\frac{b^{2s+1}}{a^{2s+1}}\right) i.$$

The term in p is the general term showing the coefficient of $\frac{b^{2p}}{a^{2p}}$ as a function of p, while the last term in the coefficient of i shows the coefficient of $\frac{b^{2s+1}}{a^{2s+1}}$ as a function of s. The several terms of the development are found by supposing
$$p = 1,\ p = 2,\ p = 3, \text{ etc.},$$
and
$$s = 0,\ s = 1,\ s = 2, \text{ etc.}$$

We also remark that, by reducing the denominators, the general terms may be expressed in the form
$$-(-1)^p \frac{1\,.\,3\,.\,5\,.\,7\dots(4p-3)}{2\,.\,4\,.\,6\,.\,8\dots 4p}\frac{b^{2p}}{a^{2p}},$$
and
$$(-1)^s \frac{1\,.\,3\,.\,5\,.\,7\dots(4s-1)}{2\,.\,4\,.\,6\,.\,8\dots 4s+2}\frac{b^{2s+1}}{a^{2s+1}}.$$

§ 337.

THE MODULUS BY CALCULATION.

1.
2.
3.
4. Modulus $= \sqrt{a^2 + b^2} = \sqrt{16+9} = 5.$

5.
6.
7.
8. Modulus $= \sqrt{a^2 + b^2} = \sqrt{9+16} = 5.$

9. Modulus $= \sqrt{a^2 + b^2} = \sqrt{25+49} = \sqrt{74}.$

10. Modulus $= \sqrt{a^2 + b^2} = \sqrt{25+36} = \sqrt{61}.$

11. Modulus $= \sqrt{a^2 + b^2} = \sqrt{25+25} = 5\sqrt{2}.$

12. Modulus $= \sqrt{a^2 + b^2} = \sqrt{25+14} = \sqrt{41}.$

13. Modulus $= \sqrt{a^2 + b^2} = \sqrt{9+4} = \sqrt{13}.$

14. Modulus $= \sqrt{a^2 + b^2} = \sqrt{9+1} = \sqrt{10}.$

15. Modulus $= \sqrt{a^2 + b^2} = \sqrt{9+1} = \sqrt{10}.$

16. Modulus $= \sqrt{a^2 + b^2} = \sqrt{9+4} = \sqrt{13}.$

§ 339.

1. $$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \frac{1}{2} + \frac{2}{2}i - \frac{1}{2} = i;$$

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^3 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

$$= i\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i;$$

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^4 = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

$$= -1;$$

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^5 = -1\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i;$$

$$\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^6=\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$
$$=-i;$$
$$\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^7=-i\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$
$$=+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i;$$
$$\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^8=\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=1.$$

2. $$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^2=\frac{3}{4}-\frac{1}{4}+\frac{\sqrt{3}}{2}i=\frac{1}{2}+\frac{\sqrt{3}}{2}i;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^3=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=i;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^4=i\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^5=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$
$$=-\frac{\sqrt{3}}{2}+\frac{1}{2}i;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^6=\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-1;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^7=-1\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-\frac{\sqrt{3}}{2}-\frac{1}{2}i;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^8=\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$
$$=-\frac{1}{2}-\frac{\sqrt{3}}{2}i;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^9=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-i;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{10}=-i\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=\frac{1}{2}-\frac{\sqrt{3}}{2}i;$$
$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{11}=\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$
$$=\frac{\sqrt{3}}{2}-\frac{1}{2}i;$$

$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{12}=\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

$$=\frac{3}{4}+\frac{1}{4}=+1.$$

3. (*a*) $\quad x^n a = a;\ a = \text{vector};$

$$n = 5;$$

$$\tfrac{1}{5}\,360^\circ = 72^\circ;$$

$$\cos 72^\circ = \tfrac{1}{4}\,(\sqrt{5}-1)\,a;$$

$$\sin 72^\circ = \tfrac{1}{4}\sqrt{(10+2\sqrt{5})}\,a;$$

$$\therefore x = \tfrac{1}{4}\left[\sqrt{5}-1+\sqrt{(10+2\sqrt{5})}\,i\right];$$

$$x^2 = \tfrac{1}{8}\left[-2-2\sqrt{5}+(\sqrt{5}-1)\sqrt{(10+2\sqrt{5})}\,i\right];$$

$$x^4 = \tfrac{1}{4}\left[(\sqrt{5}-1)-\sqrt{(10+2\sqrt{5})}\,i\right];$$

$$x = \tfrac{1}{4}\left[(\sqrt{5}-1)+\sqrt{(10+2\sqrt{5})}\,i\right];$$

$$\therefore x^5 = \tfrac{1}{16}\left[5-2\sqrt{5}+1+10+2\sqrt{5}\right].$$

Hence $\qquad x^5 = 1;$

which ought to be the case, because

$$x^5 a = a.$$

Similarly, if the vector was turned through an angle of 144 or 2×72, five such motions would make two revolutions, and thus restore it to its original position.

Taking x^2 and raising it to the 5th power, we have unity.

Also, x^3 raised to the 5th power $= x^{15} = (x^5)^3 = 1;$
x^4 " " " $= x^{20} = (x^5)^4 = 1;$
x^5, or unity, raised to the 5th power $= 1;$
whence the five roots of unity are the above values of x, x^2, x^3, x^4, and x^5.

§ 348.

1. We form the product,

$$[x-(2+\sqrt{3})]\,[x-(2-\sqrt{3})]\,(x+2)\,(x-1),$$

which we find to be

$$x^4-3x^3-5x^2+9x-2.$$

Therefore the required equation is

$$x^4-3x^3-5x^2+9x-2=0.$$

2. We form the product,

$$\{x-(3+\sqrt{5})\}\,\{x-(3-\sqrt{5})\}\,(x+3),$$

and the required equation is

$$x^3-3x^2-14x+12=0.$$

3. We form the product,

$$(x-2)(x+2)\{x-(4+\sqrt{7})\}\{x-(4-\sqrt{7})\},$$

and the required equation is

$$x^4-8x^3+5x^2+32x-36=0.$$

4. We form the product,

$$\{x-(1+\sqrt{3})\}\{x-(1-\sqrt{3})\}\{1-(1+\sqrt{5})\}$$
$$\{x-(1-\sqrt{5})\},$$

and the required equation is

$$x^4-4x^3-2x^2+12x+8=0.$$

§ 349.

1. Dividing the given equation by $x+3$, the quotient is

$$x^2-6x+4.$$

Equating this to zero, we have a quadratic equation

$$x^2-6x+4=0,$$

the roots of which are

$$3+\sqrt{5} \quad \text{and} \quad 3-\sqrt{5}.$$

Hence the roots of the original equation are

$$-3, \quad 3+\sqrt{5}, \quad \text{and} \quad 3-\sqrt{5}.$$

2. By § 343, I., algebraic sum of coefficients vanishes. Therefore $+1$ is a root.

By same Art., II., since the sum of the coefficients of the even powers of x is equal to that of the coefficients of the odd powers, then

$$-1 \text{ is a root.}$$

The absolute term is wanting, therefore

$$0 \text{ is a root.}$$

Dividing by x — each of these roots successively, we have:

$$x^5-4x^4+12x^3+4x^2-13x \div x-0, \text{ or simply } x,$$
$$=x^4-4x^3+12x^2+4x-13;$$
$$x^4-4x^3+12x^2+4x-13 \div x-1$$
$$=x^3-3x^2+9x+13;$$
$$x^3-3x^2+9x+13 \div x+1$$
$$=x^2-4x+13.$$

Equating the last quotient to zero, and solving, we have

$$x^2-4x+13=0;$$
$$x=2+3\sqrt{i}, \quad \text{and} \quad x=2-3\sqrt{i}.$$

Hence the five roots of the original equation are

$$0, \ -1, \ +1, \ 2+3\sqrt{i}, \ \text{and} \ 2-3\sqrt{i}.$$

§ 353.

1. $$F(x) = x^5 + 5x^4 + 8x^3 - 2x^2 - x + 1. \qquad (1)$$

Substitute $x + h$ for x; the result is

$$F(x+h) = (x+h)^5 + 5(x+h)^4 + 8(x+h)^3 - (2x+h)^2 - (x+h)^2 + 1. \qquad (2)$$

Developing the several terms of the second member, we have

$$(x+h)^5 = x^5 + 5x^4h + \frac{5 \cdot 4}{2}x^3h^2 + \text{etc.};$$

$$(x+h)^4 = x^4 + 4x^3h + \frac{4 \cdot 3}{2}x^2h^2 + \text{etc.};$$

$$(x+h)^3 = x^3 + 3x^2h + \frac{3 \cdot 2}{2}xh^2 + \text{etc.};$$

$$(x+h)^2 = x^2 + 2xh + h^2;$$

$$(x+h) = x + h.$$

Substituting these expressions in equation (2), we have

$$F(x+h) = x^5 + 5x^4 + 8x^3 - 2x^2 - x + 1 + (5x^4 + 5 \cdot 4x^3 + 8 \cdot 3x^2 - 2 \cdot 2x - 1)h + \text{omitted terms} \times h^2,\ h^3,\ \text{etc.}$$

The coefficient of h is the derived function required; viz., $5x^4 + 20x^3 + 24x^2 - 4x - 1.$

By inspection, the coefficient of h, or derived function, is seen to be in accordance to rule.

2. $$F(x) = x^7 - 2x^5 - 2x^3 - 2x;$$

$$F(x+h) = (x+h)^7 - 2(x+h)^5 - 2(x+h)^3 - 2(x+h).$$

$$(x+h)^7 = x^7 + 7x^6h + \text{etc.};$$

$$(x+h)^5 = x^5 + 5x^4h + \text{etc.};$$

$$(x+h)^3 = x^3 + 3x^2h + \text{etc.};$$

$$(x+h) = x + h.$$

By substitution,

$$F(x+h) = x^7 - 2x^5 - 2x^3 - 2x + \{7x^6 - 2 \cdot 5x^4 - 2 \cdot 3x^2 - 2\}h + \text{omitted terms} \times h^2,\ h^3,\ \text{etc.}$$

Therefore the derived function is

$$F'(x) = 7x^6 - 10x^4 - 6x^2 - 2,$$

the coefficient of h.

3. $$F(x) = x^6 + 12x^5 - 24x^3 + x^2 + 7;$$

$$F(x+h) = (x+h)^6 + 12(x+h)^5 - 24(x+h)^3 + (x+h)^2 + 7.$$

$$(x+h)^6 = x^6 + 6x^5h + \text{etc.};$$

$$(x+h)^5 = x^5 + 5x^4h + \text{etc.};$$

$$(x+h)^3 = x^3 + 3x^2h + \text{etc.};$$

$$(x+h)^2 = x^2 + 2xh + \text{etc.}$$

Hence, by substitution,

$$F(x+h) = x^6 + 12x^5 - 24x^3 + x^2 + 7$$
$$+ \{6x^5 + 12 \,.\, 5x^4 - 24 \,.\, 3x^2 + 2x\} h$$
$$+ \text{ omitted terms} \times h^2, h^3, \text{ etc.}$$

Therefore

$$F'(x) = 6x^5 + 60x^4 - 72x^2 + 2x.$$

4. $$F(x) = x^4 - 2ax^3 + 3b^2x^2 + a^2bx;$$
$$F(x+h) = (x+h)^4 - 2a(x+h)^3 + 3b^2(x+h)^2$$
$$+ a^2b(x+h).$$

$$(x+h)^4 = x^4 + 4x^3h + \text{etc.};$$
$$(x+h)^3 = x^3 + 3x^2h + \text{etc.};$$
$$(x+h)^2 = x^2 + 2xh + \text{etc.};$$
$$(x+h) = x + h.$$

By substitution,

$$F(x+h) = x^4 - 2ax^3 + 3b^2x^2 + a^2bx$$
$$+ \{4x^3 - 3 \,.\, 2ax^2 + 3 \,.\, 2b^2x + a^2b\} h$$
$$+ \text{ omitted terms.}$$

Therefore

$$F'(x) = 4x^3 - 6ax^2 + 6b^2x + a^2b.$$

5. $$F(x) = x^5 - 5mx^4 + 10mx^3 - 15mx^2;$$
$$F(x+h) = (x+h)^5 - 5m(x+h)^4 + 10m(x+h)^3$$
$$- 15m(x+h)^2.$$

$$(x+h)^5 = x^5 + 5x^4h + \text{etc.};$$
$$(x+h)^4 = x^4 + 4x^3h + \text{etc.};$$
$$(x+h)^3 = x^3 + 3x^2h + \text{etc.};$$
$$(x+h)^2 = x^2 + 2xh + \text{etc.}$$

By substitution,

$$F(x+h) = x^5 - 5mx^4 + 10mx^3 - 15mx^2$$
$$+ \{5x^4 - 5 \,.\, 4mx^3 + 10 \,.\, 3mx^2 - 15 \,.\, 2mx\} h$$
$$+ \text{ omitted terms.}$$

Therefore

$$F'(x) = 5x^4 - 20mx^3 + 30mx^2 - 30mx.$$

§ 359.

1. The roots of the denominator are

$$+2 \quad \text{and} \quad -2;$$

Assume

$$\frac{x+10}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}. \qquad (1)$$

Reducing the second member to a common denominator, it becomes

$$\frac{A(x-2) + B(x+2)}{(x+2)(x-2)}.$$

Since both members have the same denominators, the numerators must also be equal.

Hence

$$x + 10 = A(x-2) + B(x+2) = (A+B)x - 2A + 2B.$$

Since this must be true for all values of x, we equate the coefficients of x in each member, giving

$$A + B = 1;$$
$$-2A + 2B = 10.$$

Solving for A and B, we have

$$A = -2 \quad \text{and} \quad B = 3.$$

Substituting in (1),

$$\frac{x+10}{x^2-4} = -\frac{2}{x+2} + \frac{3}{x-2}.$$

2. By § 343, II., one of the roots of the denominator is -1. Dividing the denominator by $x+1$, we have

$$x^2 - 4 = (x+2)(x-2).$$

Hence

$$\frac{x^2+8x+4}{x^3+x^2-4x-4} = \frac{x^2+8x+4}{(x+1)(x-2)(x+2)}.$$

Assume

$$\frac{x^2+8x+4}{(x+1)(x-2)(x+2)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2}. \qquad (1)$$

Bringing second member to a common denominator and equating the numerators, we have

$$x^2 + 8x + 4 = (A+B+C)x^2 + (3B-C)x + 2B - 4A - 2C.$$

Since coefficients of like powers of x must be equal, we have

$$A + B + C = 1;$$
$$3B - C = 8;$$
$$-4A + 2B - 2C = 4.$$

Solving these equations, we find

$$A = 1, \quad B = 2, \quad C = -2.$$

Substituting these values in (1), we have

$$\frac{x^2+8x+4}{x^3+x^2-4x-4} = \frac{1}{x+1} + \frac{2}{x-2} - \frac{2}{x+2}.$$

3. Equating the denominator to zero and solving, we find the roots to be $2, -2, 1,$ and -1.

Hence

$$x^4 - 5x^2 + 4 = (x-2)(x+2)(x-1)(x+1).$$

Assume

$$\frac{2x^3 - 12x^2 - 8x + 12}{(x-2)(x+2)(x-1)(x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-1} + \frac{D}{x+1}.$$

Reducing second member to a common denominator and equating the numerators, we have

$$2x^3 - 12x^2 - 8x + 12 =$$
$$(A + B + C + D)x^3 + (2A - 2B + C - D)x^2$$
$$+ (-A - B - 4C - 4D)x - 2A + 2B - 4C + 4D.$$

Equating coefficients of like powers of x,

$$A + B + C + D = 2;$$
$$2A - 2B + C - D = -12;$$
$$-A - B - 4C - 4D = -8;$$
$$-2A + 2B - 4C + 4D = 12.$$

Solving these equations, we find

$$A = -3, \quad B = 3, \quad C = 1, \quad \text{and} \quad D = 1.$$

Hence

$$\frac{2x^3 - 12x^2 - 8x + 12}{x^4 - 5x^2 + 4} = -\frac{3}{x-2} + \frac{3}{x+2} + \frac{1}{x-1} + \frac{1}{x+1}.$$

4. $\dfrac{x}{x^2 - a^2} = \dfrac{x}{(x-a)(x+a)}.$

Assume

$$\frac{x}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{(A+B)x + a(A-B)}{(x-a)(x+a)}.$$

Hence $\qquad x = (A + B)x + (A - B)a,$

$$A + B = 1,$$
$$A - B = 0,$$

gives $\qquad A = \frac{1}{2}$ and $B = \frac{1}{2}$.

Hence $\qquad \dfrac{x}{x^2 - a^2} = \dfrac{1}{2(x-a)} + \dfrac{1}{2(x+a)}.$

5. $\dfrac{2a}{x^2 - a^2} = \dfrac{2a}{(x-a)(x+a)}.$

Assume

$$\frac{2a}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$
$$= \frac{(A+B)x + (A-B)a}{(x-a)(x+a)}.$$

Hence $\qquad 2a = (A + B)x + (A - B)a,$

$$A + B = 0,$$
$$A - B = 2,$$

giving $\qquad A = 1$ and $B = -1$.

Hence $\qquad \dfrac{2a}{x^2 - a^2} = \dfrac{1}{x-a} - \dfrac{1}{x+a}.$

6. $\dfrac{a^2b^2}{(x^2 - a^2)(x^2 - b^2)} = \dfrac{a^2b^2}{(x-a)(x+a)(x-b)(x+b)}.$

Assume

$$\frac{a^2b^2}{(x-a)(x+a)(x-b)(x+b)} = \frac{A}{x-a} + \frac{B}{x+a} + \frac{C}{x-b} + \frac{D}{x+b}.$$

Reducing to a common denominator and equating numerators, we have

$$a^2b^2 = (A+B+C+D)x^3 + (Aa - Ba + Cb - Db)x^2 + (-Ab^2 - Bb^2 - Ca^2 - Da^2)x - Aab^2 + Bab^2 - Ca^2b + Da^2b.$$

Equating coefficients of like powers, we have

$$A + B + C + D = 0;$$
$$Aa - Ba + Cb - Db = 0;$$
$$-Ab^2 - Bb^2 - Ca^2 - Da^2 = 0;$$
$$-Aab^2 + Bab^2 - Ca^2b + Da^2b = a^2b^2.$$

Solving these equations, we get

$$A = -\frac{ab^2}{2(b^2-a^2)}, \quad B = \frac{ab^2}{2(b^2-a^2)}, \quad C = \frac{a^2b}{2(b^2-a^2)},$$

and $$D = -\frac{a^2b}{2(b^2-a^2)}.$$

Hence

$$\frac{a^2b^2}{(x^2-a^2)(x^2-b^2)} = -\frac{ab^2}{2(b^2-a^2)(x-a)} + \frac{ab^2}{2(b^2-a^2)(x+a)} + \frac{a^2b}{2(b^2-a^2)(x-b)} - \frac{a^2b}{2(b^2-a^2)(x+b)}.$$

§ 360.

1. $$\frac{x+1}{x^2-2x+1} = \frac{x+1}{(x-1)^2}.$$

Assume

$$\frac{x+1}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{A_1}{x-1}. \qquad (1)$$

Reducing to a common denominator, and equating the coefficients of like powers of x, we have

$$A - A_1 = 1,$$
$$A_1 = 1;$$

hence $$A = 2.$$

Substituting in (1), we obtain the result

$$\frac{x+1}{(x-1)^2} = \frac{2}{(x-1)^2} + \frac{1}{x-1}.$$

2. $\dfrac{x-1}{(x+1)^2}$.

Assume $$\frac{x-1}{(x+1)^2} = \frac{A}{(x+1)^2} + \frac{A_1}{x+1}.$$

Reducing to a common denominator, and equating coefficients of like powers of x, we have

$$A + A_1 = -1 \quad \text{and} \quad A_1 = 1;$$

hence $$A = -2.$$

Substituting in the assumed equation, we obtain for the result

$$\frac{x-1}{(x+1)^2} = \frac{-2}{(x+1)^2} + \frac{1}{x+1}.$$

3. $\dfrac{x^2-2}{x^3-x^2-x+1} = \dfrac{x^2-2}{(x-1)^2(x+1)}$.

Assume

$$\frac{x^2-2}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{A_1}{x-1} + \frac{B}{x+1}.$$

Reducing to a common denominator, and equating coefficients of like powers of x, we have

$$A_1 + B = 1,$$
$$A - 2B = 0,$$
$$A + B - A_1 = -2;$$

from which we find

$$A = -\tfrac{1}{2}, \quad A_1 = \tfrac{5}{4}, \quad \text{and} \quad B = -\tfrac{1}{4}.$$

Therefore the given fraction is equal to

$$-\frac{1}{2(x-1)^2} + \frac{5}{4(x-1)} - \frac{1}{4(x+1)}.$$

4. $\dfrac{x+2}{x^3+x^2-x-1} = \dfrac{x+2}{(x+1)^2(x-1)}$.

Assume

$$\frac{x+2}{(x+1)^2(x-1)} = \frac{A}{(x+1)^2} + \frac{A_1}{x+1} + \frac{B}{x-1}.$$

Reducing to a common denominator, and equating coefficients,

$$A_1 + B = 0,$$
$$A + 2B = 1,$$
$$-A - A_1 + B = 2;$$

from which we find

$$A = -\tfrac{1}{2}, \quad A_1 = -\tfrac{3}{4}, \quad \text{and} \quad B = +\tfrac{3}{4}.$$

Therefore the given fraction is equal to

$$-\frac{1}{2(x+1)^2} - \frac{3}{4(x+1)} + \frac{3}{4(x-1)}.$$

§ 361.

1. $x^4 - 1$ and $x^6 - 1$.

Factoring we have

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1);$$

$$(x^6 - 1) = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1).$$

The common factors are $(x - 1)$ and $(x + 1)$.

Hence the G.C.D. is $(x - 1)(x + 1) = (x^2 - 1)$.

2. $x^3 - 1 = (x - 1)(x^2 + x + 1)$;

$x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$.

The G.C.D. is $(x - 1)$.

3. Divide the quantity of higher degree by the one of lower, as follows:

$$\begin{array}{l|l}
a^5 - 2a^4 - a^3 + 3a^2 - 2a - 15 & a^4 - a^3 - 4a^2 - a + 5 \\ \hline
a^5 - a^4 - 4a^3 - a^2 + 5a & a - 1 \\ \hline
-a^4 + 3a^3 + 4a^2 - 7a - 15 & \\
-a^4 + a^3 + 4a^2 + a - 5 & \\ \hline
2a^3 - 8a - 10 &
\end{array}$$

Using the first divisor for dividend and remainder, after removing the common factor 2, for divisor, we have

$$\begin{array}{l|l}
a^4 - a^3 - 4a^2 - a + 5 & a^3 - 4a - 5 \\ \hline
a^4 - 4a^2 - 5a & a - 1 \\ \hline
-a^3 + 4a + 5 & \\
-a^3 + 4a + 5 & \\ \hline
0 &
\end{array}$$

Hence $a^3 - 4a - 5$ is the G.C.D.

4. Multiply the first quantity by 4, and the second by 5, and divide.

$$\begin{array}{l|l}
100x^4 + 20x^3 - 4x - 4 & 100x^4 + 5x^2 - 5 \\ \hline
100x^4 + 5x^2 - 5 & 1 \\ \hline
20x^3 - 5x^2 - 4x + 1 &
\end{array}$$

$$\begin{array}{l|l}
20x^4 + x^2 - 1 & 20x^3 - 5x^2 - 4x + 1 \\ \hline
20x^4 - 5x^3 - 4x^2 + x & x \\ \hline
5x^3 + 5x^2 - x - 1 &
\end{array}$$

Multiply every term of the last expression by 4, and proceed with the division.

$$\begin{array}{l|l}
20x^3 + 20x^2 - 4x - 4 & 20x^3 - 5x^2 - 4x + 1 \\ \hline
20x^3 - 5x^2 - 4x + 1 & +1 \\ \hline
25x^2 - 5 &
\end{array}$$

Remove the factor 5 from the last remainder and use it for the next divisor.

$$\begin{array}{r|l}
20x^3 - 5x^2 - 4x + 1 & 5x^2 - 1 \\ \cline{2-2}
20x^3 \qquad\quad - 4x & 4x - 1 \\ \cline{1-1}
-5x^2 + 1 & \\
-5x^2 + 1 & \\ \cline{1-1}
0 &
\end{array}$$

Hence the G.C.D. is $5x^2 - 1$.

5.

$$\begin{array}{r|l}
a^4 + 2a^3 - 6a - 9 & a^4 + 2a^2 + 9 \\ \cline{2-2}
a^4 + 2a^2 \qquad + 9 & 1 \\ \cline{1-1}
2a^3 - 2a^2 - 6a - 18 &
\end{array}$$

Remove the factor 2 from the remainder.

$$\begin{array}{r|l}
a^4 + 2a^2 + 9 & a^3 - a^2 - 3a - 9 \\ \cline{2-2}
a^4 - a^3 - 3a^2 - 9a & a + 1 \\ \cline{1-1}
a^3 + 5a^2 + 9a + 9 & \\
a^3 - a^2 - 3a - 9 & \\ \cline{1-1}
6a^2 + 12a + 18 &
\end{array}$$

Remove the factor 6.

$$\begin{array}{r|l}
a^3 - a^2 - 3a - 9 & a^2 + 2a + 3 \\ \cline{2-2}
a^3 + 2a^2 + 3a & a - 3 \\ \cline{1-1}
-3a^2 - 6a - 9 & \\
-3a^2 - 6a - 9 & \\ \cline{1-1}
\end{array}$$

Hence $a^2 + 2a + 3$ is the G.C.D.

6.

$$\begin{array}{r|l}
m^3 + 3m^2 + 3m + 1 & m^2 - 1 \\ \cline{2-2}
m^3 \qquad - m & m + 3 \\ \cline{1-1}
3m^2 + 4m + 1 & \\
3m^2 \qquad - 3 & \\ \cline{1-1}
4m + 4 &
\end{array}$$

Remove the factor 4.

$$\begin{array}{r|l}
m^2 - 1 & m + 1 \\ \cline{2-2}
m^2 + m & m - 1 \\ \cline{1-1}
-m - 1 & \\
-m - 1 & \\ \cline{1-1}
\end{array}$$

Hence $m + 1$ is the G.C.D.

7. Multiply every term of first expression by 2.

$$\begin{array}{r|l}
2x^4 - 16x^3 + 42x^2 - 40x + 8 & 2x^3 - 12x^2 + 21x - 10 \\ \cline{2-2}
2x^4 - 12x^3 + 21x^2 - 10x & x - 2 \\ \cline{1-1}
-4x^3 + 21x^2 - 30x + 8 & \\
-4x^3 + 24x^2 - 42x + 20 & \\ \cline{1-1}
-3x^2 + 12x - 12 &
\end{array}$$

Remove the factor -3 from the remainder.

$$\begin{array}{r|l}
2x^3-12x^2+21x-10 & x^2-4x+4 \\
2x^3-\ \ 8x^2+\ \ 8x & \overline{2x-4} \\
\hline
-\ \ 4x^2+13x-10 & \\
-\ \ 4x^2+16x-16 & \\
\hline
-3x+6 &
\end{array}$$

Remove the factor -3 from the remainder.

$$\begin{array}{r|l}
x^2-4x+4 & x-2 \\
x^2-2x & \overline{x-2} \\
\hline
-2x+4 & \\
-2x+4 & \\
\hline
\end{array}$$

Hence $x-2$ is the G.C.D.

8.
$$\begin{array}{r|l}
a^7+a^6-a-1 & a^5+a^4-a-1 \\
a^7+a^6-a^3-a^2 & \overline{a^2} \\
\hline
a^3+a^2-a-1 & \\
\end{array}$$

$$\begin{array}{r|l}
a^5+a^4-a-1 & a^3+a^2-a-1 = \text{G.C.D.} \\
a^5+a^4-a^3-a^2 & \overline{a^2+1} \\
\hline
a^3+a^2-a-1 & \\
a^3+a^2-a-1 & \\
\hline
\end{array}$$

§ 362.

1.
$$\begin{array}{r|l}
x^3+3bcx+b^3-c^3 & x^3+(c-b)x^2+(b^2+bc+c^2)x \\
x^3+(c-b)x^2+(b^2+bc+c^2)x & \overline{1} \\
\hline
-(c-b)x^2-(b^2-2bc+c^2)x+b^3-c^3 &
\end{array}$$

Remove the factor $-(c-b)$ from this last expression and use it for the next divisor.

$$\begin{array}{r|l}
x^3+(c-b)x^2+(b^2+cb+c^2)x & x^2+(c-b)x+b^2+bc+c^2 \\
x^3+(c-b)x^2+(b^2+cb+c^2)x & \overline{x} \\
\hline
0 &
\end{array}$$

Hence the G.C.D. is $x^2+(c-b)x+b^2+bc+c^2$.

2.
$$\begin{array}{r|l}
x^3+3ax+a^3-1 & x^3-(a^2-2a)x+a-1 \\
x^3-(a^2-2a)x+a-1 & \overline{1} \\
\hline
(a^2+a)x+a^3-a &
\end{array}$$

Remove the factor $a(a+1)$ from this remainder.

$$\begin{array}{r|l}
x^3-(a^2-2a)x+a-1 & x+a-1 \\
x^3+(a-1)x^2 & \overline{x^2-(a-1)x+1} \\
\hline
-(a-1)x^2-(a^2-2a)x+a-1 & \\
-(a-1)x^2-(a-1)^2x & \\
\hline
x+a-1 & \\
x+a-1 & \\
\hline
\end{array}$$

Hence $x+a-1$ is the G.C.D.

3. $(a+b+c)(ab+bc+ca)-abc$
$=(b+c)a^2+(2bc+b^2+c^2)a+b^2c+c^2b.$

$$\begin{array}{l|l} (b+c)a^2+(2bc+b^2+c^2)a+b^2c+bc^2 & a^2+(b-c)a-bc \\ \underline{(b+c)a^2+(b^2-c^2)a-bc(b+c)} & \overline{(b+c)} \\ \quad (2bc+2c^2)a+2bc(b+c) & \end{array}$$

Remove the factor $2c(b+c)$ and we have left as the divisor $(a+b)$.

$$\begin{array}{l|l} a^2+(b-c)a-bc & a+b \\ \underline{a^2+ab} & \overline{a-c} \\ \quad -ca-bc & \\ \quad \underline{-ca-bc} & \end{array}$$

Hence $a+b$ is the G.C.D.

4.
$$\begin{array}{l|l} x^4+4a^4 & x^3-2a^2x+4a^3 \\ \underline{x^4-2a^2x^2+4a^3x} & \overline{x} \\ 2a^2x^2-4a^3x+4a^4 & \end{array}$$

Remove the factor $2a^2$, and we have $x^2-2ax+2a^2$.

$$\begin{array}{l|l} x^3-2a^2x+4a^3 & x^2-2ax+2a^2 \\ \underline{x^3-2ax^2+2a^2x} & \overline{x+2a} \\ 2ax^2-4a^2x+4a^3 & \\ \underline{2ax^2-4a^2x+4a^3} & \\ \qquad 0 & \end{array}$$

Giving $x^2-2ax+2a^2$ for the G.C.D.

5.
$$\begin{array}{l|l} x^3-ax^2-b^2x+ab^2 & x^2-a^2 \\ \underline{x^3-a^2x} & \overline{x-a} \\ -ax^2+a^2x-b^2x+ab^2 & \\ \underline{-ax^2+a^3} & \\ (a^2-b^2)x-a(a^2-b^2) & \end{array}$$

Remove the factor (a^2-b^2), and we have $x-a$ for the next divisor.

$$\begin{array}{l|l} x^2-a^2 & x-a \\ \underline{x^2-ax} & \overline{x+a} \\ ax-a^2 & \\ \underline{ax-a^2} & \end{array}$$

Hence $x-a$ is the G.C. D.

6.
$$\begin{array}{l|l} x^3+a^3+b^3-3abx & x^2+2ax+a^2-b^2 \\ \underline{x^3+2ax^2+a^2x-b^2x} & \overline{x-2a} \\ -2ax^2+(b^2-3ab-a^2)x+a^3+b^3 & \\ \underline{-2ax^2-4a^2x-2a^3+2ab^2} & \\ (b^2-3ab+3a^2)x+b^3-2ab^2+3a^3 & \end{array}$$

Remove the common factor $b^2 - 3ab + 3a^2$, and we have for the next divisor $x + a + b$.

$$\begin{array}{l|l} x^2 + 2ax + a^2 - b^2 & x + a + b \\ \underline{x^2 + ax + bx} & \overline{x + a - b} \\ \quad (a-b)x + a^2 - b^2 & \\ \quad \underline{(a-b)x + a^2 - b^2} & \\ \qquad\qquad 0 & \end{array}$$

Hence $x + a + b$ is the G.C.D.

7.
$$\begin{array}{l|l} x^4 - 2x^2 + 2 - \frac{2}{x^2} + \frac{1}{x^4} & x^4 - 2x^2 + \frac{2}{x^2} - \frac{1}{x^4} \\ \underline{x^4 - 2x^2 \qquad + \frac{2}{x^2} - \frac{1}{x^4}} & \overline{1} \\ \qquad\qquad 2 - \frac{4}{x^2} + \frac{2}{x^4} & \end{array}$$

Remove the factor 2, and our next division becomes:

$$\begin{array}{l|l} x^4 - 2x^2 + \frac{2}{x^2} - \frac{1}{x^4} & 1 - \frac{2}{x^2} + \frac{1}{x^4} = \text{G.C.D.} \\ \underline{x^4 - 2x^2 + 1} & \overline{x^4 - 1} \\ \qquad -1 + \frac{2}{x^2} - \frac{1}{x^4} & \\ \qquad \underline{-1 + \frac{2}{x^2} - \frac{1}{x^4}} & \end{array}$$

8.
$$\begin{array}{l|l} x^4 - x^3y + xy^3 - y^4 & x^4 + x^2y^2 + y^4 \\ \underline{x^4 + x^2y^2 + y^4} & \overline{1} \\ -x^3y - x^2y^2 + xy^3 - 2y^4 & \end{array}$$

Dividing by y, we have our next divisor.

$$\begin{array}{l|l} x^4 + x^2y^2 + y^4 & -x^3 - x^2y + xy^2 - 2y^3 \\ \underline{x^4 + x^3y - x^2y^2 + 2xy^3} & \overline{-x + y} \\ -x^3y + 2x^2y^2 - 2xy^3 + y^4 & \\ \underline{-x^3y - x^2y^2 + xy^3 - 2y^4} & \\ \qquad 3x^2y^2 - 3xy^3 + 3y^4 & \end{array}$$

Remove the factor $3y^2$.

$$\begin{array}{l|l} -x^3 - x^2y + xy^2 - 2y^3 & x^2 - xy + y^2 \\ \underline{-x^3 + x^2y - xy^2} & \overline{-x - 2y} \\ \quad -2x^2y + 2xy^2 - 2y^3 & \\ \quad \underline{-2x^2y + 2xy^2 - 2y^3} & \\ \qquad\quad 0 \qquad\; 0 \qquad\; 0 & \end{array}$$

Hence the G.C.D. sought is $x^2 - xy + y^2$.

§ 363.

1. Let y be the unknown quantity of the required equation. Then

$$y = x - 1;$$

therefore $$x = y + 1.$$

Substituting this value of x in the equation, it will become

$$(y + 1)^2 - 3(y + 1) - 4 = 0,$$

or $$y^2 - y - 6 = 0,$$

the required equation.

2. $x^3 - 3x^2 + 57x - 7 = 0.$

If y be the unknown quantity of the required equation, then

$$y = x + 5.$$

Therefore $$x = y - 5.$$

Substituting this value of x in the equation, it will become

$$(y - 5)^3 - 3(y - 5)^2 + 57(y - 5) - 7 = 0,$$

or $$y^3 - 18y^2 + 162y - 492 = 0,$$

the required equation.

§ 364.

1. $x^3 - 6x^2 + 6x - 1 = 0.$

Hence $$n = 3, \quad p_1 = -6;$$

$$h = -\frac{p_1}{n} = \frac{6}{3} = 2;$$

$$x = y + 2.$$

Making this substitution, the equation becomes

$$(y + 2)^3 - 6(x + 2)^2 + 6(x + 2) - 1 = 0,$$

or $$y^3 - 6y - 5 = 0,$$

the required equation.

2. $x^4 - 4x^3 + 3x^2 - 8 = 0.$

Hence $$n = 4, \quad p_1 = -4;$$

$$h = -\frac{p_1}{n} = \frac{4}{4} = 1.$$

Therefore $$x = y + 1.$$

Making this substitution, the equation becomes

$$(y + 1)^4 - 4(y + 1)^3 + 3(y + 1)^2 - 8 = 0,$$

or $$y^4 - 3y^2 - 2y - 8 = 0,$$

the required equation.

3. $x^5 - 5x^4 + 2x^3 + 2x^2 - 3x = 0.$

Hence $\quad n = 5, \quad p_1 = -5;$

$$h = -\frac{p_1}{n} = \frac{5}{5} = 1;$$

$$x = y + 1.$$

Substituting this value of x in the equation, we have

$(y+1)^5 - 5(y+1)^4 + 2(y+1)^3 + 2(y+1)^2 - 3(y+1) = 0,$

or $\quad y^5 - 8y^3 - 12y^2 - 8y - 3 = 0,$

the required equation.

4. $x^6 - 12x^5 + 2x^3 - x = 0.$

Hence $\quad n = 6, \quad p_1 = -12;$

$$h = -\frac{p_1}{n} = \frac{12}{6} = 2.$$

$$x = y + 2.$$

Substituting this value of x in the equation, it becomes

$$(y+2)^6 - 12(y+2)^5 + 2(y+2)^3 - (y+2) = 0,$$

or $\quad y^6 - 60y^4 - 318y^3 - 708y^2 - 745y - 306 = 0,$

the required equation.

§ 365.

1. $x^2 - 2x + 3 = 0.$

Let $\quad y = 4x, \quad x = \frac{y}{4}.$

Substituting this value of x in the given equation, and we have

$$\left(\frac{y}{4}\right)^2 - 2\left(\frac{y}{4}\right) + 3 = 0,$$

or $\quad y^2 - 8y + 48 = 0,$

the required equation.

2. $x^2 - 2x + 3 = 0.$

Let $\quad y = \frac{x}{2}.$ Therefore $x = 2y.$

Substituting in given equation, we have

$$(2y)^2 - 2(2y) + 3 = 0,$$

or $\quad 4y^2 - 4y + 3 = 0,$

the required equation.

§ 366.

1. $x^2 - 5x + 6 = 0.$

Put $\quad y = x^2.$ Therefore $x = \pm y^{\frac{1}{2}}.$

Substituting in equation $x = y^{\frac{1}{2}}$, we have

$$y + 6 - 5y^{\frac{1}{2}} = 0,$$

Substituting in given equation $x = -y^{\frac{1}{2}}$, we have

$$y + 6 + 5y^{\frac{1}{2}} = 0.$$

Since the value of y must satisfy one or the other of these equations, it must reduce their product to zero; we therefore multiply them together. The product will be

$$(y + 6)^2 - 25y = 0,$$

or $$y^2 - 13y + 36 = 0,$$

the required equation.

2. $x^3 + 12x^2 + 44x + 48 = 0.$

$$y = x^2. \quad \text{Therefore } x = \pm y^{\frac{1}{2}}.$$

Substituting $x = y^{\frac{1}{2}}$ in the given equation, we have

$$(12y + 48) + (y^{\frac{3}{2}} + 44y^{\frac{1}{2}}) = 0.$$

Making $x = -y^{\frac{1}{2}}$, we obtain

$$(12y + 48) - (y^{\frac{3}{2}} + 44y^{\frac{1}{2}}) = 0.$$

The value of y must reduce the product of these two equations to zero, for the reason given above. The product is

$$y^3 - 56y^2 + 784y - 2304 = 0,$$

the required equation.

3. $x^5 - 4x^4 - 10x^3 + 40x^2 + 9x - 36 = 0.$

$$y = x^2, \quad x = \pm y^{\frac{1}{2}}.$$

Substituting $x = y^{\frac{1}{2}}$, we have

$$(-4y^2 + 40y - 36) + (y^{\frac{5}{2}} - 10y^{\frac{3}{2}} + 9y^{\frac{1}{2}}) = 0.$$

Substituting $x = -y^{\frac{1}{2}}$, we have

$$(-4y^2 + 40y - 36) - (y^{\frac{5}{2}} - 10y^{\frac{3}{2}} + 9y^{\frac{1}{2}}) = 0.$$

The value of y must reduce the product of these two equations to zero. The product is

$$y^5 - 36y^4 + 438y^3 - 2068y^2 + 2961y - 1296 = 0,$$

the required equation.

§ 367.

1. $x^2 - 7x + 10 = 0.$

Put $$y = 3x^2, \quad x = \pm\sqrt{\frac{y}{3}}.$$

Substitute $x = \sqrt{\frac{y}{3}}$ in given equation, and we have

$$\frac{y}{3} + 10 - 7\sqrt{\frac{y}{3}} = 0.$$

Substitute $x = -\sqrt{\frac{y}{3}}$, and we have

$$\frac{y}{3} + 10 + 7\sqrt{\frac{y}{3}} = 0.$$

Multiplying these two equations, we obtain

$$y^2 - 87y + 900 = 0$$

as the required equation.

2. $x^3 - 3x^2 + 2x = 0.$

$$y = ax + b. \quad \text{Hence } x = \frac{y-b}{a}.$$

Substituting in given equation, we have

$$\left(\frac{y-b}{a}\right)^3 - 3\left(\frac{y-b}{a}\right)^2 + 2\left(\frac{y-b}{a}\right) = 0,$$

or $$y^3 - (3b + 3a)y^2 + (3b^2 + 6ab + 2a^2)y - b^3 - 3ab^2 - 2a^2b = 0,$$

the required equation.

3. $x^2 - 9x + 18 = 0.$

$$y = \tfrac{1}{3}x^2 - 3, \qquad x = \pm\sqrt{3y+9}.$$

Substituting $x = \sqrt{3y+9}$ in given equation, we have

$$3y + 9 + 18 - 9\sqrt{3y+9} = 0.$$

Substituting $x = -\sqrt{3y+9}$, we obtain

$$3y + 9 + 18 + 9\sqrt{3y+9} = 0.$$

Multiplying these equations, we obtain

$$y^2 - 9y = 0$$

as the required equation.

§ 369.

1. $Fh = 2h^5 + 23h^4 + 101h^3 + 205h^2 + 183h + 46.$

Coefficients,	2	+23	+101	+205	+183	+46
Products by − 3,		− 6	− 51	−150	−165	−54
First sums,		17	50	55	18	− 8
Second products,		− 6	− 33	− 51	− 12	
Second sums,		11	17	4	6	
Third products,		− 6	− 15	− 6		
Third sums,		5	2	− 2		
Fourth products,		− 6	+ 3			
Fourth sums,		− 1	5			
		− 6				
		− 7				

Result, $F(x-3) = 2x^5 - 7x^4 + 5x^3 - 2x^2 + 6x - 8.$

2. $$Fx = x^3 - 7x + 7,$$
when $$x = -4 + h.$$

Coefficients,	1	0	-7	$+7$
Products by -4,		-4	16	-36
First sums,		-4	9	-29
Second products,		-4	32	
Second sums,		-8	41	
		-4		
		-12		

Result, $F(-4+h) = h^3 - 12h^2 + 41h - 29.$

When $x = -3 + h$:

Coefficients,	1	0	-7	$+7$
Products by -3,		-3	9	-6
First sums,		-3	2	1
Second products,		-3	18	
Second sums,		-6	20	
		-3		
		-9		

Result, $F(-3+h) = h^3 - 9h^2 + 20h + 1.$

When $x = -2 + h$:

Coefficients,	1	0	-7	$+7$
Products by -2,		-2	4	6
First sums,		-2	-3	13
Second products,		-2	8	
		-4	5	
		-2		
		-6		

Result, $F(-2+h) = h^3 - 6h^2 + 5h + 13.$

When $x = -1 + h$:

Coefficients,	1	0	-7	$+7$
Products by -1,		-1	1	6
First sums,		-1	-6	13
Second products,		-1	2	
		-2	-4	
		-1		
		-3		

Result, $F(-1+h) = h^3 - 3h^2 - 4h + 13.$

When $x = h$:

$$F(h) = h^3 - 7h + 7.$$

When $x = 1 + h$:

Coefficients,	1	0	-7	$+7$
Products by 1,		1	1	-6
First sums,		1	-6	1
Second products,		1	2	
		2	-4	
		1		
		3		

Result, $F(1 + h) = h^3 + 3h^2 - 4h + 1.$

When $x = 2 + h$:

Coefficients,	1	0	-7	$+7$
Products by 2,		2	4	-6
First sums,		2	-3	1
Second products,		2	8	
		4	5	
		2		
		6		

Result, $F(2 + h) = h^3 + 6h^2 + 5h + 1 = 0.$

When $x = 3 + h$:

Coefficients,	1	0	-7	$+7$
Products by 3		3	9	6
First sums,		3	2	13
etc.		3	18	
		6	20	
		3		
		9		

Result, $F(3 + h) = h^3 + 9h^2 + 20h + 13.$

§ 371.

1. $$x^3 - 3x^2 + 1 = 0.$$

Let us first assume $x = 3$. We compute as follows:

Coefficients,	1	-3	0	$+1$
Products by 3,		3	0	0
Sums,		0	0	1

So $F(3) = +1$; and as all the coefficients are positive, there can be no root as great as 3.

Putting $x = -3$, the sums including the first coefficient are $1, -6, +18, -53$. These being alternately positive and negative, there is no root so small as -3.

Substituting all the integers between -3 and $+3$, we find:

$$\begin{array}{ll} F(-3) = -53 & \\ F(-2) = -19 & F(1) = -1 \\ F(-1) = -\ \ 3 & F(2) = -3 \\ F(0) \ \ = +\ \ 1 & F(3) = +1 \end{array}$$

Hence we know that there is a root between -1 and 0, another between 0 and 1, and the third root between 2 and 3.

Let us begin with the latter.

Transforming the equation to one in h, by putting $2 + h$ for x, we find the equation in h to be

$$h^3 + 3h^2 - 3 = 0. \qquad (1)$$

Substituting $h = 0.2, 0.4, 0.6, 0.8, 0.9$, we find that there is a root between 0.8 and 0.9.

If in the last equation we put $h = 0.8 + h'$, we find the transformed equation in h' to be

$$h'^3 + 5.4h'^2 + 6.72h' - 0.568 = 0. \qquad (2)$$

If we substitute different values of h' in this equation, we find that the root is between 0.07 and 0.08.

Putting $h' = 0.07 + h''$, and transforming, we find the equation in h'' to be

$$h''^3 + 5.61h''^2 + 7.4907h'' - 0.070\,797 = 0.$$

The required digit of h'' may be found by dividing 0.070 797 by 7.3907, which gives us 0.009

Now put $h'' = 0.009 + h'''$ and transform again. The resulting equation for h''' is

$$h'''^3 + 5.637h'''^2 + 7.591\,923h''' - 0.002\,925\,561 = 0.$$

The digits of x, h, h', and h'' which we have found show the true value of x to be

$$x = 2.879 + h''.$$

By continuing this process, as many figures as we please may be found.

From this point the operation may be abbreviated, by neglecting the coefficient of h^3 and using only the second decimal of the coefficient of h^2.

In this way we obtain two more figures of the root, after which the remaining figures are obtained by pure division.

The work, as we have performed it, may be arranged in the following form.

The numbers under the double lines are the coefficients of h, h', h'', etc.

1	− 3	0	+ 1	2.879 385 241
	2	− 2	− 4	
	− 1	− 2	− 3.000	
	2	2	2.432	
	1	0.00	− .568 000	
	2	3.04	497 203	
	3.0	3.04	− .070 797 000	
	.8	3.68	.067 871 439	
	3.8	6.720 0	− .002 925 561	
	.8	.382 9	2 278 084	
	4.6	7.102 9	− 647 477	
	.8	387 8	607 664	
	5.40	7.490 700	− 39 813	
	.07	.050 571	37 979	
	5.47	7.541 271	− 1834	
	7	50 652	1519	
	6.54	7.591 92|3	− 315	
	7	1 69|1	304	
	5.610	7.593 61|4	− 11	
	.009	1 69|1	8	
	5.619	7.595 3|1	− 3	
	9	4|5		
	5.628	7|.5|9|5|8		
	9			
	|5.6|37			

One root of the equation is therefore

2.879 385 241.

The second root we found to lie between 0 and 1. Without writing out the separate transformations, the work may be arranged as follows:

1	− 3.0	0.00	1.000	0.652 703 644
	.6	− 1.44	− .864	
	− 2.4	− 1.44	.136 000	
	.6	− 1.08	− .128 875	
	− 1.8	− 2.520 0	.007 125 000	
	.6	− .057 5	− .005 269 192	
	− 1.20	− 2.577 5	.001 855 808	
	.05	− 55 0	− 1 846 193	
	− 1.15	− 2.632 500	9 615	
	.05	− 2 096	− 7 914	
	− 1.10	− 2.634 596	1701	
	.05	− 2 092	− 1583	
	− 1.050	− 2.636 68 8	118	
	.002	− 73 1	− 106	
	− 1.048	− 2.637 41 9	12	
	2	73 1	− 11	
	− 1.046	− 2.6 3 8 1 5	1	
	2			
	− 1.0 44			

The second root is therefore

$$0.652\ 703\ 644.$$

To find the negative root, we change the alternate signs of our equation, thus transforming it into one having an equal positive root. The equation then is

$$x^3 + 3x^2 - 1 = 0.$$

The work, so far as it is necessary to carry it, is now arranged as follows:

1	3.0	0.00	− 1.000 0.532 088 887
	.5	1.75	.875
	3.5	1.75	− .125 000
	.5	2.00	.116 577
	4.0	3.750 0	− .008 423 000
	.5	.135 9	8 063 768
	4.50	3.885 9	− .000 359 232
	.03	.136 8	323 315
	4.53	4.022 700	− 35 917
	.03	9 184	32 332
	4.56	4.031 884	− 3585
	3	9 188	3233
	4.590	4.041 072	− 352
	.002	40	323
	4.592	4.04147	− 26
	2		28
	4.594		− 1
	2		
	4.596		

The three roots of the equation are therefore

$$x_1 = 2.879\,385\,241$$
$$x_2 = 0.662\,703\,644$$
$$x_3 = -0.532\,088\,887$$
$$\text{Sum} = 2.999\,999\,998 \text{ (check).}$$

To check the correctness of the roots as formed we take their algebraic sum, which should be equal to the coefficient of the second term with its sign changed. We see that the discrepancy is 2 units in the ninth place of decimals, which is unimportant.

2. $$x^3 - 3x + 1 = 0.$$

Let us assume $x = 2$.

Coefficients,	1	0	-3	$+1$
Products by 2,		2	4	2
Sums,		2	1	3

$F(2) = +3$; and as all the coefficients are positive, there can be no root so great as $+2$.

Putting $x = -2$, the sums, including the first coefficient, are $1, -2, 1, -1$. These being alternately positive and negative, there is no root so small as -2.

Substituting all integers between 2 and -2, we find:

$$F(2) = +3 \qquad F(-1) = +3$$
$$F(1) = -1 \qquad F(-2) = -1$$
$$F(0) = +1 \qquad F(-3) = -17$$

The roots are seen to lie between 1 and 2, 0 and 1, and between -1 and -2.

Seeking the root that lies between 1 and 2, we transform the equation to one in h, by putting $1 + h$ for x; we find the equation in h to be

$$h^3 + 3h^2 - 1 = 0.$$

Making $h = 0.2$, 0.4, 0.6, etc., we find one value of h lies between 0.5 and 0.6.

Transforming this equation by putting $h' + 0.5$ for h, we have

$$h'^3 + 4.5\,h'^2 + 3.75\,h' - 0.125 = 0.$$

By trial a root of this equation is found between 0.03 and 0.04.

Transforming this equation by substituting $0.03 + h''$ for h', and we have

$$h''^3 + 4.59\,h''^2 + 4.0227\,h'' - 0.008\,423 = 0.$$

A root of this equation is between 0.002 and 0.003.

Transforming by making $0.002 + h''' = h''$, we get

$$h'''^3 + 4.596\,h'''^2 + 4.041\,072\,h''' - 0.000\,359\,232 = 0.$$

From this point the work may be abridged by neglecting the coefficient of h'''^3, and using only to the second decimal of the coefficient of h'''^2.

We thus find two more digits of the root of the original equation. After that the remaining figures are obtained by pure division, and we may cut off one decimal from the coefficient of h''' and two from the coefficient of h'''^2 for each decimal we add to the root.

The entire work may be arranged as follows:

1	0	− 3	1	1.532 088 887
	1	1	− 2	
	1	− 2	− 1.000	
	1	2	.875	
	2	0.00	− 0.125 000	
	1	1.75	.116 577	
	3.0	1.75	− .008 423 000	
	.5	2.00	.008 063 768	
	3.5	3.750 0	− 359 232	
	.5	.135 9	323 315	
	4.0	3.885 9	− 35 917	
	.5	.136 8	32 332	
	4.50	4.022 700	− 3 585	
	.03	9 184	3 233	
	4.53	4.031 884	− 352	
	3	9 188	323	
	4.56	4.041 0|7|2	− 29	
	3	3|7	28	
	4.590	4.|0|4|1|4|4	− 1	
	.002			
	4.592			
	2			
	4.594			
	2			
	|4.5|96			

The root is therefore

1.532 088 887.

The second root lies between 0 and 1.

Without writing out the separate transformations, the work may be arranged as follows:

<table>
<tr><td>1</td><td>0.0</td><td>− 3.00</td><td>1.000</td><td>0.347 296 347</td></tr>
<tr><td></td><td>.3</td><td>.09</td><td>− 0.873</td><td></td></tr>
<tr><td></td><td>0.3</td><td>− 2.91</td><td>.127 000</td><td></td></tr>
<tr><td></td><td>.3</td><td>.18</td><td>− .107 696</td><td></td></tr>
<tr><td></td><td>.6</td><td>− 2.730 0</td><td>.019 304 000</td><td></td></tr>
<tr><td></td><td>.3</td><td>37 6</td><td>− .018 522 077</td><td></td></tr>
<tr><td></td><td>0.90</td><td>− 2.692 4</td><td>781 923</td><td></td></tr>
<tr><td></td><td>.04</td><td>39 2</td><td>− 527 713</td><td></td></tr>
<tr><td></td><td>.94</td><td>− 2.653 200</td><td>254 210</td><td></td></tr>
<tr><td></td><td>4</td><td>7 189</td><td>− 237 461</td><td></td></tr>
<tr><td></td><td>.98</td><td>− 2.646 011</td><td>16 749</td><td></td></tr>
<tr><td></td><td>4</td><td>7 238</td><td>− 15 831</td><td></td></tr>
<tr><td></td><td>1.020</td><td>− 2.638 77|3</td><td>918</td><td></td></tr>
<tr><td></td><td>.007</td><td>20|8</td><td>− 792</td><td></td></tr>
<tr><td></td><td>1.027</td><td>− 2.638 56|5</td><td>126</td><td></td></tr>
<tr><td></td><td>7</td><td>20|8</td><td>− 106</td><td></td></tr>
<tr><td></td><td>1.034</td><td>− 2.638 3|5|7</td><td>20</td><td></td></tr>
<tr><td></td><td>7</td><td>9 4</td><td>− 18</td><td></td></tr>
<tr><td></td><td>|1.0|41</td><td>− 2.|6|3|8|4|5</td><td>+ 2</td><td></td></tr>
</table>

The second root is therefore

$$0.347\ 296\ 347.$$

To find the negative root, we change alternate signs of the given equation, and it becomes

$$x^3 - 3x - 1 = 0.$$

The work may be arranged as follows:

1	0	− 3	− 1	|1.879 385 242
	1	1	− 2	
	1	− 2	− 3.000	
	1	2	2.432	
	2	0.00	− .568 000	
	1	3.04	.497 203	
	3.0	3.04	− .070 797 000	
	.8	3.68	.067 871 439	
	3.8	6.720 0	− .002 925 561	
	.8	.382 9	2 278 084	
	4.6	7.102 9	− 647 477	
	.8	.387 8	607 661	
	5.40	7.490 700	− 39 816	
	.07	50 571	37 979	
	5.47	7.541 271	− 1 837	
	.07	50 652	1 519	
	5.54	7.591 92|3	− 318	
	.07	1 69|1	304	
	5.610	7.593 61|4	− 14	
	.009	1 69|1	15	
	5.619	7.595 3|1	+ 1	
	.009	4|5		
	5.628	7.|5|9|5|7|6		
	.009			
	|5.6|37			

Hence the three roots are

$$
\begin{aligned}
x_1 &= 1.532\,088\,887 \\
x_2 &= 0\;347\,296\,347 \\
x_3 &= -1.879\,385\,242 \\
\text{Sum} &= -0.000\,000\,008 \text{ (check).}
\end{aligned}
$$

We see that the results agree to eight places of decimals.

3. $$x^4 - 4x^2 + 2 = 0.$$

Let us assume $x = 2$.

Coefficients,	1	0	-4	0	$+2$
Products by 2,		2	4	0	0
Sums,		2	0	0	2

$F(2) = +2$; and as all the coefficients are positive, there can be no root so great as 2.

Putting $x = -3$, the sums, including the first coefficient, are alternately positive and negative; hence there is no root so small as -3.

Substituting all integers between 2 and -3, we find,

$$F(2) = +2 \qquad F(-1) = -1$$
$$F(1) = -1 \qquad F(-2) = +2$$
$$F(0) = +2 \qquad F(-3) = +47$$

The roots are seen to lie between 1 and 2, 0 and 1, -1 and 0, and -2 and -1.

Seeking the root that lies between 1 and 2, we transform the given equation to one in h by putting $1 + h$ for x. We find the equation h to be

$$h^4 + 4h^3 + 2h^2 - 4h - 1 = 0.$$

Substituting in this equation $h = 0.2, 0.4, 0.6, 0.8$, etc., we find that there is a root between 0.8 and 0.9.

Transforming this equation by putting $0.8 + h'$ for h, we have

$$h'^4 + 7.2h'^3 + 15.44h'^2 + 8.928h' - 0.4624 = 0.$$

By trial, a root of this equation is between 0.04 and 0.05.

Putting $0.04 + h''$ for h', we get

$$h''^4 + 7.36h''^3 + 16.3136h''^2 + 10.198\,016h'' - 0.08\,011\,246 = 0.$$

A root of this equation is formed between 0.007 and 0.008.

Transforming, we have

$$h'''^4 + 7.388h'''^3 + 16.468\,454h'''^2 + 10.427\,489\,692h''' - 0.007\,924\,634\,719 = 0.$$

The value of x is therefore $1.847 + h'''$.

The value of h''', and hence the true value of x, may be found by continuing this process.

But we may shorten the process by cutting off one decimal of the coefficient of h, two from the coefficient of h^2, and three from the coefficient of h^3, for each additional decimal of the root.

The entire work may be arranged as follows:

1	0	− 4	0	+ 2	1.847 759 065 022 6
	1	1	− 3	− 3	
	1	− 3	− 3	− 1.000 0	
	1	2	− 1	.537 6	
	2	− 1	− 4.000	− .462 400 00	
	1	3	4.672	.382 287 36	
	3	2.00	.672	− .080 112 640 000	
	1	3.84	8.256	72 188 005 281	
	4.0	5.84	8.928 000	− 7 924 634 719	
	.8	4.48	629 184	7 307 314 861	
	4.8	10.32	9.557 184	− 617 319 858	
	8	5.12	640 832	522 569 030	
	5.6	15.440 0	10.198 016 000	− 94 750 828	
	8	289 6	114 556 183	94 071 179	
	6.4	15.729 6	10.312 572 183	−679 649	
	8	291 2	114 917 509	627 150	
	7.20	16.020 8	10.427 489 69¦2	− 52 499	
	04	292 8	11 531 53 8	52 263	
	7.24	16.313 600	10.439 021 23	− 236	
	4	51 569	11 535 16	209	
	7.28	16.365 169	10.450 556 3¦9	− 27	
	4	51 618	824 2	21	
	7.32	16.416 787	10.451 380 6	− 6	
	4	51 667	824 2	6	
	7.360	16.468 4¦54	10.452 204¦8	0	
	7	5 1 72	148 4		
	7.367	16.473 6¦26	10.452 353¦2		
	7	5 1 72	148 4		
	7.374	16.478 7¦98	1¦0¦.4¦5¦2¦5¦0¦2		
	7	5 1 72			
	7.381	¦16.¦48¦39¦70			
	7				
	7.¦388				

Therefore the root is 1.847 759 065 022 6.

The work for the second root, which is found between 0 and 1, may be arranged as follows:

				0.765 366 864 73
1	0.0	− 4.00	0.000	2.000 0
	.7	.49	− 2.457	− 1.719 9
	7	− 3.51	− 2.457	.280 100 00
	7	.98	− 1.771	− .256 878 24
	1.4	− 2.53	− 4.228 000	.033 221 760 000
	7	1.47	− 53 304	− .021 633 459 375
	2.1	− 1.060 0	− 4.281 304	1 588 300 625
	7	171 6	− 42 792	− 1 298 807 346
	2.80	− .888 4	− 4.324 096 000	289 493 279
	6	175 2	− 2 595 875	− 259 771 981
	2.86	− .713 2	− 4.326 691 875	29 721 298
	6	178 8	− 2 519 625	− 25 977 391
	2.92	− .534 400	− 4.329 211 50¦0	3 743 907
	6	15 225	− 146 31 9	− 3 463 654
	2.98	− .519 175	− 4.329 357 81¦9	280 253
	6	15 250	− 146 04 4	− 259 774
	3.040	− .503 925	− 4.329 503 8¦6	20 479
	5	15 275	− 29 1 5	− 17 318
	3.045	− .488 6¦50	− 4.329 533 0¦1	3 161
	5	9 18	− 29 1 5	− 3 031
	3.050	− .487 7¦32	− 4.329 562¦2	130
	5	9 18	− 2 9	− 130
	3.055	− .486 8¦14	− 4.329 565¦1	
	5	9 18	− 2 9	
	3¦.060	− .48¦58¦96	− 4.3¦2¦9¦5¦6¦8	

The second root is therefore

0.765 366 864 73.

To find the third and fourth roots, which are negative, we change the alternate signs of the given equation, and we thus transform it into an equation which has equal positive roots.

The given equation written in full is

$$x^4 \pm 0x^3 - 4x^2 \pm 0x + 2 = 0.$$

Changing alternate signs it becomes

$$x^4 \mp 0x^3 - 4x^2 \mp 0x + 2 = 0,$$

or

$$x^4 - 4x^2 + 2 = 0,$$

which is identically the given equation; the positive roots of which we have already found.

Hence the negative roots are equal to the positive ones, and are

$$-1.847\,759\,065\,023\,6$$

and

$$-0.765\,366\,864\,730\,0$$

This equality of the negative and positive roots may be shown by solving the given equation as a quadratic in x^2.

4.

$$x^2 + x - 1 = 0.$$

We assume $x = 1$.

Coefficients,	1	1	-1
Products by 1,		1	2
Sums,		2	1

$F(1) = +1$; and as all the coefficients are positive, there can be no root so great as 1.

Putting $x = -2$, the sums, including the first coefficient, are alternately positive and negative; hence there is no root so small as -2.

Substituting all integers between 1 and -2, we find

$$F(1) = +1 \qquad F(-1) = -1$$
$$F(0) = -1 \qquad F(-2) = +1$$

Showing that there is one root between 0 and 1, and another between -1 and -2.

Seeking the root that lies between 0 and 1 first, we substitute $0 + h$ for x, and we have

$$h^2 + h - 1 = 0.$$

By trial we find that it is between 0.6 and 0.7.

Transforming by putting $0.6 + h'$ for h, we get

$$h'^2 + 2.2\,h - 0.04 = 0.$$

This equation has a root between 0.01 and 0.02.

Substituting $0.01 + h''$ for h', we have

$$h''^2 + 2.22\,h'' - 0.0179 = 0.$$

A root of this equation is of the form $0.008 + h'''$.

Transforming it to one in h''', we have

$$h'''^2 + 2.236\,h''' - 0.000\,076 = 0,$$

which has a root between 0.000 03 and 0.000 04.

After transforming it becomes

$$h^{iv} + 2.236\,06\,h^{iv} - 0.000\,008\,919\,1 = 0.$$

From this point we get the remaining decimals of the root of the given equation by simple division, as follows:

2.|2|3|6|0|6	− 0.0 000 089 191	|39 887
	67 082	
	− 22 109	
	20 125	
	− 1 984	
	1 789	
	− 195	
	179	
	− 16	
	16	

Hence the root is 0.618 033 988 7

Changing the alternate signs, and thus transforming the equation to one having an equal positive root, it becomes $x^2 - x - 1 = 0$.

The entire work may be arranged as follows:

1	− 1	− 1	|1.618 033 988 7
	1	0	
	0	− 1.00	
	1	.96	
	1.0	− .040 0	
	6	22 1	
	1.6	− .017 900	
	6	17 824	
	2.20	− .0 000 760 000	
	.01	670 809	
	2.21	− 89 191	
	1	67 082	
	2.220	− 22 109	
	.008	20 125	
	2.228	− 1 984	
	8	1 789	
	2.236 00	− 195	
	3	179	
	2.236 03	− 16	
	3	16	
	2.|2|3|6|0|6		

The negative root is therefore

− 1.618 033 988 7

5. Take the general equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots p_{n-1}x + p_n = 0. \quad (1)$$

Changing the alternate signs it becomes

$$x^n - p_1x^{n-1} + p_2x^{n-2} - \dots \mp p_{n-1}x \pm p_n = 0, \quad (2)$$

or

$$-x^n + p_1x^{n-1} - p_2x^{n-2} + \dots \pm p_{n-1}x \mp p_n = 0. \quad (3)$$

Equations (2) and (3) are equivalent, since changing all the signs of one makes it identical with the other.

Suppose that a is a root of equation (1), then will $-a$ be a root of (2) and (3).

Substituting a for x in equation (1), we have

$$a^n + p_1a^{n-1} + p_2a^{n-2} + \dots + p_{n-1}a + p_n = 0. \quad (4)$$

If n is even, and we substitute $-a$ for x in (2), we have

$$a^n + p_1a^{n-1} + p_2a^{n-2} + \dots + p_{n-1}a + p_n = 0; \quad (5)$$

but if n is odd, we would have

$$-a^n - p_1a^{n-1} - p_2a^{n-2} - \dots - p_{n-1}a - p_n = 0.$$

Changing all the signs of this equation, it becomes identical with equation (5).

Equations (4) and (5) are identical.

Hence if a satisfies equation (1), $-a$ satisfies equation (2).

That is, if the alternate signs of the coefficients of an equation are changed, the sign of the root will be changed.

APPENDIX.

ALGEBRAIC PROBLEMS.

When the College Algebra to which the present volume is a key is used to instruct beginners in the subject, oral teaching and exercises may be found necessary in the first book. The following exercises are intended as suggestions to the teacher in this respect, which he can follow out at pleasure, until he finds that the pupil readily comprehends the relative meaning of positive and negative quantities and operations.

§ 17.

At this point the pupil should, before going farther, be exercised in algebraic addition by questions like the following:

1. A thermometer stands 10° below zero; it then rises 5°, falls 3°, rises 22°, and falls 6°. Find how high it will be by adding positive and negative quantities separately and taking their algebraic sum.

2. Explain by the scale of numbers the algebraic sum of $+3$ and -2. Of $+2$ and -3. $+2$ and -5. -2 and $+5$.

3. In what different ways do we form the sums $+5$ and -3? -5 and $+3$?

Show that the difference between these operations is merely one of order of arrangement, the quantity first written being that which we measure from the zero-point. It should also be understood at this point that the algebraic sum of these quantities is the same in whatever order they are put together.

4. A man started from his house, walked 5 miles east; then 2 miles west; then 3 miles east; then 7 miles west. How far

was he then west of his house? How far was he east of his house in algebraic language? Explain the difference in the way the signs are used when we take east as positive, and when we take west as positive.

5. A man purchased \$20 worth of goods from his grocer and paid him \$15. At another time he purchased \$10 worth and paid \$12. Express what he still owed the grocer in algebraic language, assuming as positive his debt to the grocer. Then, changing all the signs and assuming what the grocer owed him as positive, express the result.

6. A balloon with a buoyancy of 500 pounds had two men in its car weighing 150 pounds each. Express the combined weight of the balloon and men algebraically. Then express algebraically the force with which the balloon was drawn upwards.

7. When east is positive, what is the algebraic sign of 5 miles west? What the algebraic sign of 5 miles east? Express these distances when west is positive.

ALGEBRAIC SUBTRACTION.

§ 21.

1. What is the difference between the following quantities, no regard being paid to the sign of the difference:

$+1$ and -1? 4 and -6? 6 and -4?

Which is the greater, $-4-2$ or $-4+2$? $1-1$ or $+1-1$? $5-3$ or $3-5$?

§ 26.

1. What does -2 hours afternoon mean?
" -3 " "

Which is the later -2 hours or -3 hours?

2. A railway runs east and west through the city of Springfield. Assuming west to be positive, what do you mean when we say the train runs -30 miles an hour?

3. Assuming a train to be passing through Springfield at the rate of $+20$ miles an hour, how far will it be in $+2$

hours? What would be its distance at the time -2 hours expressed algebraically?

To be explained that because -2 hours means 2 hours before the time that it passed through Springfield, therefore the train at that time would be east of Springfield, corresponding to the algebraic theorem that -2 hours $\times 20 = -40$ miles.

4. The train runs $+2$ hours at the rate of -20 miles an hour. Explain what this means, and the result.

5. The train passed through Springfield at noon, running -20 miles an hour; what was its distance at the time -2 hours? Obtain this by the multiplication; -2 times -20.

§ 34.

1. What is the value of $3 - b$ when
$b = 7$? $b = 5$? $b = 3$? $b = 1$? $b = -1$? $b = -4$?
2. What is the value of $3 + b$ when
$b = 1$? $b = 0$? $b = -1$? $b = -2$? $b = -3$?
3. What is the value of $a + b$ when
$b = -3$? $b = -2$? $b = -1$? $b = 0$? $b = +1$?
4. What is the value of $a - b$ when
$b = +1$? $b = 0$? $b = -1$? $b = -2$? $b = -3$?
5. When b is positive, which is the greater, $a - b$ or $a + b$? Which is the greater when b is negative?

ANSWERS

TO

SUPPLEMENTARY EXERCISES.

---

15. 1. $+3$. 2. $+6$. 3. $+3$. 4. -13.

21. *a.* -18. *b.* -17. *c.* $+49$.

34. 1. $16, -16; -6, +6; -13, +13$; etc.

2. $-14, 16; -11, 13$; etc.

56. 5. Total height $9x+9t-9y$.

6. $6(h-k)+15(m+n)$. 7. x east of y, $k-h+x-y$.

8. B, $x+y-m$ west of A; C, $6m-x$ west of B.

9. $A=a-5d+5y+4g-4z$;
$B=x+5d-5y-4g+4z$.

10. $A=r-m+q$; $C=r+n-p$; $D=n-q+p$.

11. $A=a-b+c-d+2x$; $B=2b-2x$;
$C=2x-2b+2c$.

12. $A=a-c+d$; $B=b-a+c$; $C=c-b+a$.

13. 1st side $=2a$; 2d side $=2b-2a$;
4th side $=-a+b-c+d$.

14. A west of B $m-p$ 1st day;
$2m-2p+k+x$ 2d day;
$4m-4p+3k+3x$ 4th day.

61. 1. $5m+h$. 2. $-2b$. 3. $4b$. 5. $2c-5d$.

6. $m-14n-2h+a$. 8. $12a-x$. 9. 0.

10. $-x-4a$. 12. $10m-6n$. 13. $4b-5h$.

74. 2. $mx+2mny+2bmy$. 3. $a^2n^2-a^4n^4$.

4. $h+h^2+h^3+h^4$. 5. $x-x^2+x^3-x^4$.

7. $px-qx^2+rx^3-x^4$. 8. $a^2x^3+abx^2+acx+ad$.

9. $a^2x^3-abx^2-acx-ad$. 10. $p^2x^3+p^3x^2+p^4x+p^5$.

11. $m^2x^4-m^3x^3-m^4x^2-m^5x$. 13. $2a^2my-2b^2my$.

14. $a^2-a^2b+a^2bc-abcd.$ 15. $a^2b-a^2bc+a^2c^2d-ac^2d^2.$
16. $2ab-2xy.$ 17. $2my+2nx.$

76. 1. $a^2x^3+(a^3-a^2)x^2-a^3x+a^4+a^3+a^2.$
2. $-x^3+(a-1)x^2+(1-a-a^2)x+1-a+a^2-a^3.$
3. $-max^3-(ma^2-na)x^2-(na^3-ma^2+na)x$
$+ma^4+na^3-ma^2+na.$
4. $x^4-dx^3+cx^2-bx+a.$ 5. $(a^4-b^4)x^2+(a^4-b^4)x.$

Miscellaneous Exercises in Factoring.

1. $x(ax-2b+c).$ 3. $x(a^2-4+1)-y(2ca+1).$
6. $x^2(t-pqx+px^{n-2}).$ 11. $(x^2-2)(x^2+2).$
14. $(2x-3y)(2x+3y).$ 15. $\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right).$
20. $\left(x^m+\frac{1}{2}\right)\left(x^m+\frac{1}{2}\right).$ 22. $(4a^{2n}-1)(4a^{2n}+1).$
25. $\left(\frac{x}{2}-\frac{1}{3y}\right)\left(\frac{x}{2}-\frac{1}{3y}\right).$ 29. $6ab(2a-3b)^2.$
31. $\left(2+\frac{1}{8}\right)\left(2-\frac{1}{8}\right).$ 37. $\left(\frac{5x^2}{4}-y\right)\left(\frac{5x^2}{4}+y\right).$
42. $\frac{xy}{6}(1-z)(1+z).$ 50. $(a-y^n)^2.$ 57. $x^4(2x^2-1)^2.$

97. 1. $a+2x.$ 2. $3a^4+3a^2b^2+3b^4.$ 4. $a^2+6a+9.$
5. $24a^2-2ab-35b^2.$ 6. $a^2+\frac{3}{4}.$ 7. $11ab^2.$
8. $20a^2-80ab+15b^2.$ 9. $a^2-7ab.$ 10. $x+y.$
12. $5a^2-6ab-2b^2.$ 14. $4a^{\frac{2}{3}}-a^{\frac{1}{6}}.$ 15. $3a^{-1}+2.$
17. $x^2y^{-2}-1+x^{-2}y^2.$ 18. $x^{2n}+x^ny^n+y^{2n}.$
20. $x^2+2ax+\frac{1}{3}a^2.$ 22. $x^2+y^2.$ 23. $4x^2+2x-3.$
25. $ac-3bc+2ab.$ 26. $a+1-c-b$; R, $4bc-4b.$
27. $8x+9y-6z.$ 29. $3x-4y-6z$; R, $3xz-3x^2.$
30. $(2x+3y)(4x^2+6xy+9y^2).$ 32. $ab+ac-bc.$
33. $a^2-ab-ac+b^2+c^2-bc.$ 34. $x^2-px+q.$
35. $a^2-5a+6.$ 37. $3a^2-5b^2+3c^2.$
38. $y^3+3y^2+3yx^2+x^3.$ 39. $2ax+ab+c.$
41. $6x+6y-20.$ 43. $ab^m+cb^n+x.$
44. $x^8-x^6y^2-x^2y^6+y^8.$ 46. $5x^4-4x^3+3x^2-2x+1.$

49. $ay^2 - ab + by$. 50. $a + b$. 53. $1 - 6z + 9z^2$.

110. 1. $\frac{b+a}{b-a}$. 4. m. 5. $\frac{ac}{b}$. 6. $\frac{a}{bc}$. 7. $\frac{1}{ac} + ab + bc$.

9. $\frac{(r+s)^2}{2r}$. 10. $\frac{r^3+r}{r^2+r-1}$. 13. $\frac{1}{b^2} - ab$.

129. 1. $m - n$. 2. abc. 3. $\frac{a}{2b-a}$. 4. ab.

9. $\frac{70ab - 3ac}{320c}$. 10. $\frac{ac - 3a^2bc}{c - ad}$. 11. $\frac{ab-1}{bc+d}$.

12. $\frac{3n-6}{4}$. 13. $\frac{3a^2 + ma - 3ac}{c - a + m}$. 14. $\frac{a^2+a}{1-2a}$.

15. $-\frac{11}{12}$. 16. 3. 18. $\frac{ac}{b}$. 19. $\frac{abc}{bc + ac + ab}$.

20. $\frac{a^6 + a^4}{a^3 - a^2 + a - 1}$. 21. $\frac{1}{2(b-a)}$.

22. $\frac{a^2b^3 - b^2 + a^2 - a^3b^2}{b-a}$. 23. $m^2 - n^2$. 24. $\frac{m+n}{p}$.

25. $\frac{m+b}{1+mb}$. 26. $\frac{2mq - p}{m + 2m + 1}$. 27. $\frac{bc + ab - b^2}{a}$.

28. $\frac{m+n}{2}$. 30. b. 31. $-\frac{n}{m}$. 32. $\frac{n(2m+1)}{m(2m-1)}$.

33. ah. 34. $\frac{a(1+2h)}{1-2h}$.

137 to 140. 1. $\frac{na}{m+n}$; $\frac{ma}{m+n}$. 2. $\frac{a+qb}{p+q}$; $\frac{pb-a}{p+q}$. 3. p; 0.

4. 0; 0. 5. $\frac{2b^2 - 6a^2 + d}{3a}$; $\frac{d - b^2 + 3a^2}{3b}$.

6. $\frac{ab-2a}{b^2-1}$; $\frac{a-2ab}{1-b^2}$. 7. $\frac{2a}{c}$; $-\frac{2b}{c}$. 8. 4; $\frac{4}{9}$.

9. $\frac{as}{a+b}$; $\frac{bs}{a+b}$. 10. $\frac{ag+bf}{cg+bd}$; $\frac{ag+bf}{dg-cf}$.

11. $\frac{b^2-a^2}{bd-ac}$; $\frac{b^2-a^2}{bc-ad}$. 12. $2b + a$; $2a + b$.

13. $\frac{c-b}{a}$; $\frac{c-a}{m}$. 15. $\frac{(m+n)r}{2mn}$; $\frac{(n-m)r}{2mn}$.

16. $\frac{a}{a-b}$; $\frac{b}{a+b}$. 17. 2; 2.

Equations with Three or More Unknown Quantities.

1. $\dfrac{maq}{m(p+q)+nq}$; $\dfrac{naq}{m(p+q)+nq}$; $\dfrac{map}{mp+(m+n)q}$.

2. $\dfrac{as}{b+c+a}$; $\dfrac{bs}{b+c-a}$; $\dfrac{cs}{a+b+c}$.

3. $a+\dfrac{b}{2}$; $-\dfrac{b}{2}$; $2a+\dfrac{b}{2}$.

4. $\dfrac{ms}{m+n+p}$; $\dfrac{ns}{m+n+p}$; $\dfrac{ps}{m+n+p}$.

5. $\dfrac{24}{19}$; $\dfrac{36}{19}$; $\dfrac{54}{19}$. 6. $\dfrac{2}{3}$; -7; $36\frac{1}{3}$.

7. $\dfrac{m}{2}(a+b+c)$; $\dfrac{m}{2}(a+b-c)$; $\dfrac{m}{2}(a+c-b)$.

8. $\dfrac{mbu}{m+1}$; $\dfrac{bu}{m+1}$; $u(a-b)$. 9. $-\dfrac{1}{4}$; $-\dfrac{7}{4}$; $-\dfrac{3}{4}$.

10. $\dfrac{na-b+a-c}{n^2+n-2}$; $\dfrac{nb-c-a+b}{n^2+n-2}$; $\dfrac{nc-a-b+c}{n^2+n-2}$.

11. 0; 4; 4; 4. 13. $\dfrac{d(d-b)(c+b)-d(d^2-b^2)}{a(a-b)(c+b)-a(a^2-b^2)}$.

14. $\dfrac{2abc}{ac+bc-ac}$; $\dfrac{2abc}{ab+bc-ac}$; $\dfrac{2abc}{ab+ac-bc}$.

15. 6; 5; 8. 16. $\dfrac{13}{25}$; $\dfrac{17}{25}$; $\dfrac{19}{25}$.

17. $\dfrac{a-b+c-d}{2}$; $\dfrac{a+b-c+d}{2}$ ....

18. $Z=\dfrac{a^2c^3-b^2c^3-a^3bc+a^4c}{abc+a^2b+ac^2+bc^2}$.

19. $y=\dfrac{ab(c-b)-bc(c-b)}{c(bc^2-ac^2+a^2c+ab^2-a^2b)}$.

Problems with One Unknown Quantity.

1. \$70,000. 2. (*a*) $n-2m$; (*b*) $\dfrac{mn}{m-2n}$.

3. 13 years ago. 5. Time $=\dfrac{nk}{m+n}$; distance $=\dfrac{mnk}{m+n}$.

6. $\frac{hk}{n} + k$. 7. 8 miles. 9. Current = 2.704 miles.

11. 384; 162; 118; 104. 12. 36. 14. 10; 50.

15. $\frac{9}{19}l$; $\frac{10}{19}l$. 17. $\frac{mn}{m+n}$. 18. $2\frac{1}{7}$ years.

19. 19.82 years. 21. $36m$.

23. $\frac{c+d}{m-n}$ = length; $\frac{nc+nd}{m-n} + c$ = cost.

24. $30,000. 25. 45. 26. $n - 1$. 28. 24.

30. 346\frac{2}{3}$; 96\frac{2}{3}$; 76\frac{2}{3}$.

31. $\frac{2c - 4m + 2a}{3}$ = share of third. 33. $\frac{cn}{c-b} - n$.

34. 20. 36. 30. 37. $\frac{cd}{c-d+a}$. 38. 151\frac{2}{3}$, B's share.

39. $\frac{3a - m - 3n}{5 + 3m}$ = C's share. 40. 8 days. 42. 36.

44. 36. 45. $\frac{ma + bc - bm}{c - m}$ = wife's age.

46. $400 = A's share. 48. 80. 49. 12. 51. 12; 3.
53. 8; 6. 55. 4; 5. 58. 6; 24. 60. 400. 61. 6.
63. 100 lbs. 64. 38 working days. 66. $1600.
67. 100. 69. 40. 72. $12\frac{1}{2}$.
70. First son, $600; youngest daughter, $660.
73. The plasterers receive $150. 75. 32. 77. $800.
78. 162 artillerymen. 80. 112. 83. $14\frac{2}{5}$ hours.
85. 24. 86. By boat, 36. 87. 14; 28. 88. 36.
90. 9.15; 3.85. 91. $4\frac{4}{5}$. 93. 2. 94. 6; 5. 96. 91.

97. $5550. 98. $\frac{6}{8}$. 99. $29\frac{1}{5}$; $10\frac{4}{5}$. 101. $4000.

102. 1968 feet. 104. $4000, value of house. 105. 50; 15.
107. 200; 150. 108. 150. 111. $1562. 113. $660.
114. $25. 116. $7\frac{1}{5}$. 118. 18 feet broad.
120. 27. 121. $500; $400. 123. Two 25-cent pieces.
124. 18; 4. 125. $750. 127. 16; 34; 105.
130. 432. 132. $1200. 134. 82. 135. 56.
137. $844.75. 138. A's share, $110.

140. 1. 65. 2. $152\frac{2}{7}$. 4. 69,000,000 miles.

5. 4 boys; 3 girls. 7. 172; 19. 8. 15; 63.

10. $\frac{m}{m+1}$; $\frac{m^2}{m+1}$. 12. 5; 11. 14. 12; 18.

15. $2\frac{1}{2}$; $12\frac{1}{2}$. 16. Rate of boats, $8\frac{1}{3}$; $11\frac{1}{3}$; current, $1\frac{2}{3}$.

18. Distance, $205\frac{1}{3}$ miles.

21. $\frac{1}{2}(n+\sqrt{n^2+4n})$; $\frac{1}{2}(n-\sqrt{n^2+4n})$.

23. 100; 150.

25. Any number whose unit's digit is twice the ten's [digit.

26. 56. 29. $\frac{18}{52}$.

31. 20; 40. 33. Indeterminate. 34. 84; 184.

35. 20; 30. 36. 20; 30. 37. 36.

39. 30; $17\frac{1}{7}$; $13\frac{1}{3}$ — 120. 41. 12; 20.

42. 1. 43. 12; 70; 86. 45. 12; 36.

47. 235. 49. 204. 50. 7; 3. 51. 80; 40.

52. 21; 82. 53. 1; 2; 2. 54. 3; 6; 9.

55. 5 per cent, $1200. 58. 15; 24.

60. 19,500 feet. 61. 5 and 7; 75 cts.; $1.00.

63. 35; 40. 65. $30\frac{2}{9}$; $37\frac{7}{9}$. 67. 283. 68. 18; 36.

70. 15; cost, 30 cts. 73. 12 cts.; $1.20. 77. 44.

78. $5000; $7200; $$2666\frac{2}{3}$. 81. 92 cts; $1.48.

82. $2; $3.

164. 1. 27; 36; 63. 3. .0180; .0120; .0144.

6. 2550; 3400. 8. 18. 10. 4; 6.

13. $\frac{cn-dm}{c-d}$. 18. $4\frac{5}{6}$; $24\frac{1}{6}$. 20. 8 inches.

22. 16 parts gold, 14 silver. 24. 14. 27. $14\frac{4}{11}$.

32. $\frac{4}{13}N$; $\frac{6}{13}N$; $\frac{3}{13}N$. 33. $12\frac{1}{2}$; $16\frac{1}{4}$. 35. 7; 10.

38. 3 litres. 41. $24a$. 44. $45; $82.50; $157.50.

46. 4. 48. 56; 98. 52. $2m$; $3m$. 53. $75m$; $50m$.

56. $\frac{bx}{2x-b}$. 59. $(\beta)\ d = 2b$; $(\delta)\ d = b^2x + 2b^2x^2$. 61. $25\frac{1}{5}$ inches.

64. 32 inches. 66. 4; 1; 3. 68. 28; 12.

69. 3; $\frac{1}{3}$. 71. 450. 72. $11\frac{3}{7}$; $4\frac{4}{7}$.

74. Rate, 25, 20; distance, 150 yds. 75. $4\frac{4}{5}$; $6\frac{1}{5}$; 5.

179. 1. $x^{\frac{1}{4}}$. 2. $3x^{\frac{1}{10}}b^{\frac{1}{15}}c^{\frac{1}{4}}$. 4. $x^{\frac{2}{3}}y^{\frac{1}{4}}z$. 5. $a^{-1}b$.
7. $-x^{-6}y^{-2}z^{-7}$. 9. $8x^4yz+5z^6-3x^3y^{-6}$.
12. $7x^3y^3z^2+4x^{\frac{2}{3}}z^{\frac{13}{6}}-3y^{-2}z$. 14. $x^{\frac{2}{3}}-b^2+x^{\frac{1}{4}}$.
16. $x^{\frac{m}{n}-2}-3x^{-2-n}$. 17. $4x^{\frac{n}{m}+\frac{1}{2}}-2x^{\frac{3}{2}}$.

182. 3. $(mn^2p^3)^{\frac{1}{6}}$. 5. $(18)^{\frac{1}{4}}$. 7. $(h^4r^8s^3)^{\frac{1}{6}}$. 10. $\sqrt{4050}$
12. $(c^9b^2)^{\frac{1}{6}}$. 14. $(h^{-3}b^2)^{\frac{1}{6}}$. 16. $\left(8\,.\,12^{-\frac{2}{3}}\right)^{\frac{1}{6}}$.

183. 2. $9\sqrt{3}$. 4. $2\sqrt{2}$. 6. $-13\sqrt{3}$. 7. $4\sqrt{a}$.
9. $(3a^2b+5ab)\sqrt{2ab}$. 11. $(8a^3b-5ab^2c+6b)(4abc)^{\frac{1}{4}}$.
12. $(3a^2b-2a^{-1}b^2+a^{m+3}+c)(2a^m)^{\frac{1}{3}}$.
14. $(a+b)(x+y)^{\frac{1}{4}}$. 15. $\left(\frac{p+q}{p-q}\right)^{\frac{1}{4}}$.
16. $\left(\frac{mp}{m^2+2mn+n^2}\right)^{\frac{1}{2}}$.

184. 1. $c^2-ac\sqrt{r}+bc\sqrt{r}-abr$.
3. $amn-am^2\sqrt{a-z}+n^2\sqrt{a-z}-mn(a-z)$.
4. $-Z$. 6. $p^{\frac{1}{2}}(p-1)^{\frac{1}{2}}-2p^{\frac{3}{2}}-p^{\frac{1}{2}}$.
8. $a^2+2ay^{\frac{1}{2}}-x+y$. 9. $\frac{m^2-n^2}{mn}$. 11. $\frac{r^3-a^3}{ar}$.
12. $\frac{2x}{(c^2-x^2)^{\frac{1}{2}}}$.

195 1. 5 or -11. 2. 4 or -8. 3. 17 or -3.
to 4. $\frac{1}{2}(37\pm\sqrt{89})$. 5. 1 or -7. 6. 2 or 6.

198. 8. $\frac{1}{2}$ or -1. 10. a or b. 12. 8 or $-\frac{9}{4}$.
14. 12 or $-\frac{1}{2}$. 15. 9 or $\frac{15}{13}$. 17. $\frac{1}{8}(21\pm\sqrt{1641})$.
18. 3 or 8. 20. -2 or $\frac{1}{3}$. 21. 3 or -11.
23. 9 or 2. 24. 13 or -6. 25. -4 or $-\frac{6}{7}$.

27. 3 or -3. 28. $a \pm b$. 30. 36 or $-\frac{1}{36}$.

32. 4 or $2\frac{2}{5}$. 33. 6 or $\frac{3}{5}$. 35. 28 or 9.

36. 2 or $\frac{12}{5}$. 38. $2 \pm 3\sqrt{2}$. 40. 3 or $-\frac{1}{2}$.

42. 4 or $-\frac{14}{3}$. 43. 8 or $2\frac{1}{2}$. 44. 4 or $-\frac{5}{3}$

46. 6 or $\frac{1}{2}$. 47. -1 or 4. 49. 6 or $\frac{19}{7}$.

50. ∞. 52. $\frac{25}{36}$. 53. $1 \pm 2\sqrt{2}$. 54. 1 or $\frac{1}{21}$.

55. 4 or $-12\frac{3}{5}$. 56. 3. 58. 8 or $\frac{72}{17}$. 59. $7\frac{1}{5}$.

61. 13 or 193. 62. $\frac{1}{2}$. 63. $\frac{1}{2}\sqrt{m^2 - 8}$.

65. 1 or 4. 66. ± 3. 68. 9 or $-16\frac{1}{3}$.

69. 1. 70. $\frac{1}{4}\left(\frac{2 - 2a + a}{1 - a}\right)^2$. 71. 1.

73. ∞ or -1. 74. 6 or $\frac{1}{5}$. 76. 2 or $-\frac{2}{3}$.

78. 5 or 2. 80. $\pm 2\sqrt{-1}$. 82. -4 or $-\frac{1}{3}$.

84. -8 or 3. 86. $3 \pm \sqrt{5}$. 87. $\frac{2}{3}$ or $-3\frac{1}{2}$.

89. 3 or $-\frac{87}{10}$ 90. $\frac{1}{2}$ or $\frac{103}{148}$. 92. $\frac{1}{2}(1 \pm 3)\sqrt{13}$.

94. -8 or 5. 95. 3 or -6. 97. ± 2.

98. 6 or $-4\frac{3}{4}$. 100. 1 or $-3\frac{4}{15}$.

202. 1. Add and subtract b^2. 2. $(x - 3a)(x + a)$.

4. $(x - y)(x + 9y)$. 6. $(2x - 1)(x - 3)$.

8. $(2a + 3b)(2a - 5b)$. 10. $(6a - b)(a + 5b)$.

12. $(a^2 + 9b^2 - 9ab)(a^2 + 9b^2 + 9ab)$.

14. $(a + b)(a + 3b)$. 16. $x^4(x^2 + 8y^2)$.

18. $3(2x + y)(2x + 3y)$. 19. $(4a^2 - b)(a^2 - 9b)$

207. 1. $x = 4$ or 3; $y = 3$ or 4. 3. $x = \pm 5$; $y = \pm 2$.

5. $x = \sqrt{\dfrac{a^2 + b^2}{2}}$; $y =$ the same.

6. $x = 7$ or 21; $y = 21$ or 7.

8. $x = 2$ or 6; $y = 6$ or 2. 10. $x = \pm 10$; $y = \pm 15$.

12. $x = 2$ or $-\dfrac{1}{3}$; $y = 14$ or $2\frac{1}{3}$.

13. $x = 15$ or 30; $y = 0$ or 15.

15. $x = 4$ or $\dfrac{1}{2}$; $y = 3$ or $2\frac{2}{3}$.

17. $x = 2$ or -1; $y = 3$ or -2.

19. $x = \pm 2$; $y = 3$. 20. $x = \dfrac{7}{\sqrt{3}}$; $y = \dfrac{2}{\sqrt{3}}$.

22. $x = \pm 2.38$; $y = \pm 6.04$.

24. $x = 3$ or 5; $y = 5$ or 3.

26. $x = 4$ or -2; $y = 2$ or -4.

28. $x = 3$ or 24; $y = 2$ or $\dfrac{1}{2}$.

29. $x = 2$ or 3; $y = 3$ or 2. 30. $x = 3$ or 9.

32. $y = \pm 3$. 34. $y = \pm 3$. 36. $x = \pm 6$.

38 $x = \dfrac{3 + \sqrt{5}}{2}$ or $\dfrac{1 + \sqrt{5}}{2}$. 39. $y = \pm 1$.

40. $x = \pm 3$. 42. $x = 4$ or $-\dfrac{4}{3}$.

44. $y = 4$ or $14\frac{25}{36}$. 46. $x = -6$ or 5. 47. $y = 2$.

48. $x = \pm 2$. 50. $y = 2$. 51. $x = 6$ or $4\frac{6}{7}$.

52. $x = 4$ or 15. 54. $x = \sqrt{2}$ or ± 7.

55. $y = \pm 2$. 57. $x = 3$ or 24 58. $x = 9$ or 4.

60. $x = 2$. 61. $x = 7$. 62. $y = 3$ or 4.

64. $x = -6$ or 8.

Three or More Unknown Quantities.

1. $x = 4$ or $3\frac{3}{23}$. $y = 6$ or $7\frac{2}{3}$; $z = 8$ or -12.

3. $x = 8$ or 6; $z = 9$ or 4.

4. $y = 2\frac{5}{6}$ or 5; $z = 9$ or 4.

5. $x = 8$ or 2; $z = 2$ or 8. 7. $x = 2$; $y = 3$.

8. $x = -6$; $z = 4$. 10. $x = 4$; $y = 3$.

12. $x = 4$; $v = -1$. 13. $u = \dfrac{173 \mp \sqrt{3767}}{36}$.

14. $u = 3\frac{1}{2}$; $v = 3\frac{1}{2}$; $y = 7$.

Problems Leading to Quadratic Equations.

1. Time 5 or 6 years; rate, 6 or 5 per cent.
3. 3; 4; 5. 4. 12 and 15 or -10 and -18.
6. 27 and 8. 8. 15 cts. 9. 6; 7; 8.
10. 3 and 4. 12. 4. 14. 12 and 15.
16. 6 and 9. 17. $\dfrac{8}{6}$ and $\dfrac{6}{4}$.
19. 5 per cent and 6 per cent. 20. 20; 30. 21. 10.
23. A's, \$100; B's, \$150. 25. $\sqrt{58} - 4 =$ B's rate.
26. 3 and 25. 28. 9 lbs. at 15 cts. or 15 lbs. at 9 cts.
30. A's, \$329.91+. 32. 20.
34. $4\sqrt{10}$; $3\sqrt{10}$. 35. 2 : 1. 37. 23.
38. 4; 6; 8; 10; 12. 40. $\dfrac{1}{288}$; $\dfrac{7}{2592}$.
42. 16; 20. 44. 3; 5. 46. 6; 8. 48. $\sqrt[4]{5}$.
50. 6. 51. 12. 53. 6; 1.
55. 10 per cent. 57. 30; 36.
59. 8; 4. 60. 18; 12. 62. 7; 4. 64. \$44,000.

Progressions.

1. $\left(\dfrac{n}{2} - 1\right)(l + a)$. 3. $-n$; $+n$.

5. $np \pm \dfrac{n^2(n-1)}{2}$. 8. $m' = 2(m - n) + 1$.

9. $3a$; a; $-a$. 11. $\dfrac{k}{3}$.

13. $l = a + (b - a)n - 1$. 16. $2n + n^2$.

18. $d = \dfrac{2(k-1)}{n}$ 19. $i[n - m + 2(i-1)h]$.

21. $a = \dfrac{h^7}{h^6 + k^6}$. 22. $r = \sqrt{2}$.

23. $r = \sqrt{3}$; $a = 2$. 25. $r = \sqrt[n]{6}$ or $\sqrt[n]{-4}$.

28. \$64. 29. \$16,000.

34. $-\frac{32}{5}$; $+\frac{48}{5}$; $-\frac{72}{5}$; . . . $+\frac{243}{5}$.

36. $\frac{1}{8}$; $\frac{1}{4}$; . . . 16. 38. $\frac{16}{3}$. 40. $\frac{n}{n-m}$.

42. $\frac{r^2+1}{r^2-r+1}$. 45. $\frac{r}{(1-r)(1+ar)}$. 47. 1.

50. $n(n+1)$. 51. $\frac{r}{(1-ar)(1-br)}$.

Functional Notation.

5. $S(2n) = (r^n + 1)\, S(n)$.

6. $\frac{n-s}{s+1}$; $\frac{n-s}{s}$; . . . $\frac{n-s-2}{S}$.

17. $C_t = h \cdot \frac{s^t - 1}{s-1} + s^t C_0$.

18. $\frac{3}{4}$; $\frac{11}{16}$; . . . $C_t = \frac{1}{3}\left[2 + \left(\frac{1}{4}\right)\right]$.

19. $C_{2n} = \frac{1}{3}\left\{\left[2 + \left(\frac{1}{4}\right)^{2n}\right]a + \left[2 - 2\left(\frac{1}{4}\right)^{2n}\right]b\right\}$.

Permutations and Combinations.

1. 6! 3. 144. 6. 18 letters. 7. $2^n \cdot n!$

9. 96. 11. $(n-1)!$ 13. 1080. 14. 105.

15. 903. 17. $\frac{\overline{n-k} \cdot \overline{n-k-1} \ldots \overline{n-s+1}}{\overline{s-k}!}$.

19. 75; 200; 15. 21. 15,504. 23. 36.

25. 10,080. 27. 5040. 28. 64 . 63 . 62 . 61.

29. 27,720. 31. $n = \frac{r}{2}$. 35. $C^n_r\ C^n_s (r+s)!$

37. $\frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2}$.

Indeterminate Coefficients.

1. $1 + (1+n)x + (1+n)x^2$ . . .

3. $1 + (m-n)x - n(m-n)x^2 + n^2(m-n)x^3$ . . .

5. $1 + \frac{a-1}{a^2}x - \frac{1}{a^2}x^2 - \frac{a-1}{a^4}x^3$ . . .

6. $a - x + \frac{x^2}{a} - \frac{x^3}{a^2} \ldots$

7. $a - \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} - \frac{x^5}{a^4} \ldots$

8. $x+(1+b)x^2+(1+b+b^2)x^3+(1+b+b^2+b^3)x^4 \ldots$

9. $x + \left(\frac{1}{c} + c\right)x^2 + \left(\frac{1}{c^2} + 1 + c^2\right)x^3 \ldots$

10. $x^2 - x^4 + x^6 - x^8 \ldots$

12. $\frac{1}{a^2} + \frac{x}{a^3} - \frac{x^3}{a^5} - \frac{x^4}{a^6} \ldots$

Products of Series.

1. $1 + x^2 + x^4 + x^6 \ldots x^n + x^{n+1}$.

3. $y^2 - \frac{y^4}{4} + \frac{y^6}{9} \ldots (-)^{n-1}\frac{y^{2n}}{n^2}$.

4. $1 + \left(a + \frac{1}{a}\right)x + \left(a^2 + 1 + \frac{1}{a^2}\right)x^2 \ldots$

5. $1 - \left(a + \frac{1}{a}\right)x + \left(a^2 + 1 + \frac{1}{a^2}\right)x^2 - \ldots$

6. $1 + 4x + 10x^2 \ldots \frac{n \,.\, n+1 \,.\, n+2}{1 \,.\, 2 \,.\, 3}x^n$.

Figurative Numbers.

1. $\frac{(3n+1)(n+1)}{8}$. 2. 108. 4. 364.

6. 120. 7. 1155. 9. 12. 10. 650.

11. 280. 17. $(i+1)\left\{ab + \frac{ih(b+a)}{2} + \frac{i(2i+1)h^2}{6}\right\}$.

18. $n \,.\, ab + \frac{n \,.\, n-1}{2}(bh + ak) + \frac{n(n-1)(2n-1)}{6}hk$.

20. $n^2a - \frac{n(n-1)(4n+1)}{2}$. 24. 4 . 6 . 7. 8.

25. $\left(\frac{8}{4}\right)$ 26. $\frac{n \,.\, n+1 \,.\, n+2 \,.\, n+3 \,.\, n+4}{5}$.

28. $\beta + n\alpha$; $\gamma + \frac{n \,.\, n-1}{2}\beta \ldots \left(\frac{n}{l+1}\right)\alpha$.

29. $a + nb + \frac{n \,.\, n-1}{2}c$.

Summation of Series.

1. $\dfrac{1+n-x}{(1-x)^2}$. 2. $\dfrac{1}{(1-x)^3}$. 4. $\dfrac{3}{4}$. 6. $\dfrac{1}{4}$.

8. $\dfrac{11}{96}$. 10. $\dfrac{a(1-r)}{(1-ar)(a-r)}$. 12. $\dfrac{n}{(n-1)^2}$.

14. $\dfrac{n\ .\ n+1\ .\ n+5}{6}$. 15. $\dfrac{n\ .\ n+1,\ 2n+7}{6}$.

19. $a\dfrac{n}{(n-1)^2}$.

Limits.

1. 1. 2. ∞. 3. 0. 5. $\dfrac{2}{3a}$. 6. $-\dfrac{2}{a}$.

8. $\dfrac{a}{b}$. 10. $\dfrac{1}{3}$. 12. $\dfrac{1}{m+1}$. 13. $\dfrac{2}{9}$.

Binomial Theorem.

1. $1+x+x^2+x^3\ \ldots\ .$ 2. $1+2x+3x^2\ \ldots$

4. $1+nx+\left(\dfrac{n}{2}\right)x^2+\left(\dfrac{n}{3}\right)x^3\ \ldots$

6. $1-\dfrac{mx}{m}+\dfrac{mx(mx-1)}{1\ .\ 2\ .\ m^2}-\dfrac{mx}{3}\dfrac{1}{m^3}\ \ldots$

8. $a^{\frac{1}{2}}-\dfrac{1}{2}a^{-\frac{1}{2}}b-\dfrac{a^{-\frac{3}{2}}b^2}{2^3}-\dfrac{3a^{-\frac{5}{2}}b^3}{2^3\ .\ 2\ .\ 3}\ \ldots$

10. $a^{-\frac{1}{2}}+\dfrac{a^{-\frac{3}{2}}b}{2}+\dfrac{3\ .\ a^{-\frac{5}{2}}}{2^3}+\dfrac{5a^{-\frac{7}{2}}}{2^4}$.

12. $x^6-6x^4+15x^2-20+\dfrac{15}{x^2}-\dfrac{6}{x^4}+\dfrac{1}{x^6}$.

13. $x^{2n}-2nx^{2n-2}+\dfrac{2n\ .\ 2n-1}{2}x^{2n-4}$, etc.

15. $x^{2n+1}-(2n+1)x^{2n-1}+\dfrac{2n+1}{2}x^{2n-3}\ \ldots$

17. $1+nx+\left\{n+\left(\dfrac{n}{2}\right)\right\}x^2+\left\{2\left(\dfrac{n}{2}\right)+\left(\dfrac{n}{3}\right)\right\}x^3\ \ldots$

19. $1 - nx + \left\{n + \frac{n \cdot n+1}{2}\right\}x^2$
$- \left\{n \cdot n+1 + \frac{n \cdot n+1 \cdot h+2}{2 \cdot 3}\right\}x^3 \ldots$

21. $\frac{1 \cdot 2 \cdot 3}{3!} + \frac{2 \cdot 3 \cdot 4}{3!}x + \frac{3 \cdot 4 \cdot 5}{3!}x^2$
$- \ldots \frac{i \cdot i+1 \cdot i+2}{3!}x^{i-1} \ldots$

25. $(1-h)^{-\frac{1}{2}}$. 27. $\left(1 - \frac{h}{2}\right)^{-\frac{1}{2}}$ 28. $\left(1 - \frac{1}{3}\right)^{-\frac{1}{2}}$.

30. $(1+x)^{-2n}$; $\binom{-2n}{i-1}x^{i-1}$.

32. $(1-x)^{-2n}$; $\binom{-2n}{i-1}(-x)^{i-1}$.

34. $(1+x)^{-n}$; $\frac{-n}{i-1}x^{i-1}$. 36. 1. 38. 4^m.

40. $\left(1 - \frac{1}{3}\right)^{-0}$. 41. $\left(1 - \frac{x}{n}\right)^{-n}$.

Exponential Theorem.

1. Develop e^x and multiply by e^a; develop e^{a+x}, and prove identity of coefficients of $x \cdot x^2 \ldots x^n$.

2. $1 + mx + \frac{m^2x^2}{2} + \frac{m^3x^3}{1 \cdot 23} \ldots \frac{m^6x^6}{6!}$.

4. Coefficient of $x^n = \frac{41}{\lfloor n}(1-1)^n = 0$.

Logarithms.

1. $n \log a + \frac{n \cdot n-1}{2}\log r$. 3. $x = b^{ay}$.

4. $x = \frac{1}{2}b^y$. 6. $x = \frac{b^{my}}{a}$. 8. $x = b^{\frac{my}{n}}$.

9. $y = x$. 11. $x = b^{\frac{2 \log y}{\log ma}}$.

12. 1. 13. $a^{\log c}$ or $c^{\log a}$.

ALGEBRA